Essays on Arbitrage Pricing Theory and Systemic Risk Modeling

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Kapitel 1

Introduction and Summary of Research Findings

1.1 Introduction

This thesis aims to shed light on three important problems in the broad field of *Mathematical Finance.* In the **first** problem we address the papers Lin and Chang (2009, 2010), which derive a VIX futures and option pricing theory by modeling the S&P 500 index as a stochastic volatility process with jumps in asset return and volatility. We thereby show that the formula derived in Lin and Chang (2009, 2010) is not correct and also empirically leads to mispricing in VIX futures and options.

The **second** problem considered in this thesis is the Fundamental Theorem of Asset Pricing (FTAP). The FTAP is a cornerstone in mathematical finance, but the recent financial crisis has led to a flourishing discussion on its weaknesses. In particular, the FTAP assumes that the real-world probabilities of events are known and does not address the case when there is uncertainty about these probabilities. In this thesis we derive a robust FTAP, where we do not fix a real-world probability measure but allow for uncertainty in it, by only assuming to be given a class of real-world probability measures, which does not have to be dominated. The resulting martingale measures will be discrete and their support, as well as their conditional support, satisfy certain conditions.

The **third** paper considers a network of banks, where banks are linked to each other through overnight lending. Additionally, banks are allowed to obtain liquidity by entering an overnight repurchase agreements and also have longer-term, unsecured borrowings. The balance sheet of every bank is modeled and various shocks to the systems are added, such as a shock to haircuts/margins in repo transactions, a shock in asset values or simply a default of a bank. Numerical evaluations are guided in order to examine the evolvement of contagion.

1.2 Summary of Research Findings

In the following, we give a brief summary of the content of the three research papers.

1.2.1 A Remark on Lin and Chang's Paper 'Consistent Modeling of S&P 500 and VIX Derivatives'

Since VIX options have become exchange-listed product for volatility trading, the need for a flexible model to consistently price S&P 500 options and VIX options is of great interest to the traders of these products. Attempts for such a model have firstly been made by Zhang and Zhu (2006), where the stochastic volatility model of Heston (1993) is used to describe the S&P 500. In the paper Lin and Chang (2009, 2010), the authors model the S&P 500 by a stochastic volatility process with asset return and volatility jumps. However, the formula they derive to consistently price index options and options on the VIX cannot be a correct solution of their pricing equation. We formally prove that in their framework, the characteristic function of their pricing equation cannot be exponentially affine, as proposed by them. Furthermore, by comparing it to the Heston (1993) stochastic volatility model we demonstrate that their formula can also not serve as a good approximation.

1.2.2 A Robust Fundamental Theorem of Asset Pricing with Discrete Martingale Measures

Under the assumption that the events of zero probability are known, the Fundamental Theorem of Asset Pricing (FTAP) gives an equivalence of the absence of arbitrage and the existence of certain martingale measures. In reality, one faces the problem that the zero sets are unknown which then might lead to uncertainty about the real-world probability measure, i.e. we end up with a class of probability measures, which come into question as real-world measures, such that this class is even non-dominated. This framework is a breakpoint for the classic FTAP and one need to rethink its validity in this more general case. A first promising result in this framework was obtained by Bouchard and Nutz (2014), which we also follow closely to obtain our results. Given such a class of probability measures $\mathcal{P}$ we define a robust arbitrage to be, roughly speaking, a trading strategy that leads to no loss $\mathbb{P}$-almost surely under every $\mathbb{P} \in \mathcal{P}$ and it leads to a positive gain with positive probability under at least one measure $\mathbb{P} \in \mathcal{P}$. Clearly, if the set $\mathcal{P}$ contains only one measure our framework reduces to the classic FTAP-framework. Then, we continue to show that the absence of a robust arbitrage is equivalent to the existence of a discrete martingale measure, whose conditional support satisfies certain conditions. More precisely, the conditional support of the martingale measure has to coincide with the joint conditional support of the measures in $\mathcal{P}$. Contrary to the classic FTAP, and also in contrast with Bouchard and Nutz (2014), we drop here completely the condition of absolute continuity of measures and only put restrictions on the support of measures. This leads to more flexibility and also is more promising for obtaining future results.

1.2.3 Simulating Negative Feed-Back Effects in Financial Systems

After the systemic events in the recent financial crisis, practitioners as well as regulators have realized the importance of models which capture the effects of interconnectedness in financial systems. Liquidity in the overnight interbank market played a major roje in the 2007-2008 financial crisis and a good summary of further key effects is given in Brunnermeier (2009). In our model, we do a daily simulation of the balance sheets of banks, which are linked through overnight interbank lending. Due to this interconnectedness, the default of a bank can cause the immediate default of other banks and we call this effect a *direct spill-over effect.* Additionally, banks are allowed to borrow through a repurchase agreement (repo), thereby facing the risk of higher margins and haircuts. On a short time frame (e.g. 2 weeks), the interbank unsecured lending rate can also heavily impact the performance of a bank. These latter effects are termed *indirect spill-over effects,* since they usually take place as a consequence of the direct effect. Another very important indirect effect, for which we account for in our model, are fire sales.

Although there is no clear empirical evidence of its occurrence during the crisis, its existence is widely believed and regardless of it occurrence, a complete model must incorporate also fire sales. Under this setup, we show that considering only direct effects in a model leads to a severe underestimation of the extent of a crisis. Furthermore, our model adds to the literature of contagion models by incorporating several effects which have previously eithe not been considered at all, or only been considered separately. Altogether, this provides a very reasonable toolbox which can be fed with real data of banking networks in order to make policy conclusions.

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Kapitel 2

A Remark on Lin and Chang's Paper 'Consistent Modeling of S&P 500 and VIX Derivatives'

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Leippold, M., Cheng, J., Ibraimi, M., and Zhang, J. E. (2012). A remark on Lin's and Chang's pager 'Consistent modelling of S&P 500 and VIX derivatives'. *Journal of Economic Dynamics and Control*, 36(5):716-718.

Abstract: Lin and Chang (2009, 2010) establish a VIX futures and option pricing theory when modeling S&P 500 index by using a stochastic volatility process with asset return and volatility jumps. In this note, we prove that Lin and Chang's formula is not an exact solution of their pricing equation. More generally, we show that the characteristic function of their pricing equation cannot be exponentially affine, as proposed by them. Furthermore, their formula cannot serve as a reasonable approximation. Using the Heston (1993) model as a special case, we demonstrate that Lin and Chang formula misprices VIX futures and options in general and the error can become substantially large.

2.1 Introduction

VIX options have become very successful exchange-listed products for volatility trading. The bid-ask spread of VIX options market is large due to the fact that a commonly accepted VIX option pricing model is not available yet. Hence, developing a tractable VIX option pricing model is important for the healthy growth of the new market. Yet, as the VIX index is directly linked to the implied volatility of the S&P 500 index and hence to index options, a VIX option pricing model needs to provide enough flexibility to jointly price in a consistent manner options on the S&P 500 as well as on the VIX index.

The first attempt to express the price of VIX futures was made in Zhang and Zhu (2006), where the stochastic volatility model of Heston (1993) is used to describe S&P 500. They developed a simple theoretical model for VIX futures prices and tested the model using the actual futures price on one particular day. Dotsis et al. (2007) studied the continuous-time models of the volatility indices. Zhu and Zhu and Zhang (2007) further derived a no-arbitrage pricing model for VIX futures using the time-dependent long-term mean level in the volatility model. Lin (2007) incorporates simultaneous jumps in both asset return and volatility processes. Sepp (2008) used the square root stochastic variance model with jumps in the variance process to describe the evolution of S&P500 volatility, and showed how to price and hedge VIX futures and VIX options in this model. Albanese et al. (2009) studied volatility derivatives by using spectral methods. Zhang and Huang (2010) studied the CBOE S&P500 three-month variance futures market, and showed a linear dependence between the price of fixed time-to-maturity variance futures and the VIX by using a simple mean-reverting stochastic model for the S&P500 index. Lu and Zhu (2010) studied the variance term structure using VIX futures market. Zhang et al. (2010) studied VIX futures market by using a stochastic volatility model with stochastic long-term mean level. Some other recent studies about the VIX and its derivatives include Chen et al. (2010), Dupoyet et al. (2011), Hilal et al. (2011), Konstantinidi and Skiadopoulos (2011), Cont and Kokholm (2011), Shu and Zhang (2012), and Zhu and Lian (2012) among others. Carr and Lee (2009) provided an interesting review on volatility derivatives market.

Lin and Chang (2009, 2010) establish a VIX futures and option pricing theory when modeling S&P 500 index by using a stochastic volatility process with asset return and volatility jumps. Hence, their model seems to suggest a pricing framework which is both tractable and flexible enough to consistently price index options and options on the VIX. However, we show that Lin and Chang's (2009, 2010) formula published in both papers is not an exact solution of their pricing equation. More generally, we formally prove that the characteristic function of their pricing equation cannot be exponentially affine, as proposed by them. One could still argue that their formula provides a reasonable approximation for an option pricing formula that, given their general setup, does not allow for a closed-form solution. However, by using a reduced-form specification of their model, we find that their formula can also not serve as an approximation. In particular, we use the simple setup of the Heston (1993) stochastic volatility model and we demonstrate that Lin and Chang formula misprices VIX futures and options in general and the error could be substantially large. We further point out that for the simultaneous pricing of index and VIX options, an exact formula has actually been provided by Sepp (2008) under the assumption of a stochastic volatility process with volatility jumps but no jumps in asset return.[1]

This note is structured as follows. In the next section, we briefly review some general results on affine jump diffusions and their characteristic function. In Section 2.3, we present the main result of Lin and Chang (2009, 2010). In Section 2.4, we provide a formal proof showing that the result of Lin and Chang cannot be correct and we also show that their formula cannot serve as an appropriate approximation of the true pricing formula. Section 2.5 concludes.

2.2 Affine Jump Diffusion

Let $\mathcal{X} \subset \mathbb{R}$ be a closed set with non-empty interior. Throughout this note we assume that for every $x \in \mathcal{X}$ there exists a solution $X = X^x$ of the one-dimensional stochastic differential equation

$$dX_t = \mu(X_t)dt + \sigma(X_t)dB_t + dJ_t, \quad X(0) = x, \tag{2.1}$$

where J is a pure-jump process with jump arrival intensity $\Lambda(S_t)$ at time t for some $\Lambda : \mathbb{R} \to [0, \infty)$. Jump sizes $Z_1, Z_2, \ldots$ are *iid* and independent of the Brownian motion B, which is defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$.

[1] After completing this paper, the authors became aware of another paper by Lian and Zhu (2011), which raises serious doubts about the correctness of the Lin and Chang pricing formula.

Definition 1: We call X *affine* if the $\mathcal{F}_t$-conditional characteristic function of X_T is exponential affine in X_t, for all $t \leq T$. That is, there exist $\mathbb{C}$-valued functions $\phi(T-t,z)$ and $\psi(T-t,z)$ with jointly continuous t-derivatives such that $X = X^x$ satisfies

$$\mathbb{E}[e^{zX_T} \mid \mathcal{F}_t] = \mathbb{E}_t[e^{zX_T}] = e^{\phi(T-t,z)+\psi(T-t,z)X_t}, \tag{2.2}$$

for all $z \in i\mathbb{R}, t \leq T$ and $x \in \mathcal{X}$.

Before we explain why the calculations of Lin and Chang are wrong, we briefly elaborate on an example in which the problem of determining a characteristic function is reduced to solving a system of ordinary differential equations (ODE). The same strategy is followed by Lin and Chang (2010) to find a solution for the characteristic function of the logarithm of the VIX squared and therefore deserves some attention.

Example 1: We consider the calculation of the following expectation

$$f(X_t, t) = \mathbb{E}(e^{X_T} \mid X_t) \tag{2.3}$$

under the assumption of affine dependence of μ and σ^2 on X, i.e., we assume $\mu(x) = a + bx, \sigma(x)^2 = cx$ and $\lambda(x) = l_o + l_1 x$ for some coefficients $a, b, c, l_0, l_1 \in \mathbb{R}$. If f has two continuous derivatives, the application of Itô's formula for jump diffusions gives

$$\begin{aligned} f(X_t,t) =& f(X_0,t) + \int_0^t \gamma(X_{s-},s)ds + \int_0^t f_x(X_{s-},s)dB_s \\ &+ \sum_{0<s\leq t} [f(X_s,s) - f(X_{s-},s)], \end{aligned} \tag{2.4}$$

where

$$\gamma(x,t) = f_t(x,t) + f_x(x,t)\mu(x) + \frac{1}{2} f_{xx}(x,t)\sigma(x)^2. \tag{2.5}$$

Under some technical regularity conditions we can show that $f(X_t,t)$ is a martingale.

Therefore, we get

$$\begin{aligned}0 =& f_t(x,t) + f_x(x,t)\mu(x) + \frac{1}{2} f_{xx}(x,t)\sigma(x)^2 \\ &+ \mathbb{E}[\lambda(x+Z_i)f(x+Z_i,t) - \lambda(x)f(x,t)].\end{aligned} \tag{2.6}$$

To solve the above partial differential equation (PDE) we conjecture a solution of the form $f(x,t) = e^{\alpha(T-t)+\beta(T-t)x}$. Substituting this conjectured solution into (2.6) we obtain

$$\begin{aligned}&e^{\alpha(T-t)+\beta(T-t)x}\big(-\alpha'(s) - \beta'(s)x + \beta(s)(a+bx) \\ &+ \frac{1}{2}\beta(s)^2c^2x + l_0[\mathbb{E}(e^{\beta(s)Z_i}) - 1] + l_1(\mathbb{E}[Z_ie^{\beta(s)Z_i}] + \mathbb{E}[(e^{\beta(s)Z_i} - 1)]x\big) = 0.\end{aligned} \tag{2.7}$$

Dividing by $e^{\alpha(T-t)+\beta(T-t)x}$ and collecting terms in x, we get

$$u(s)x + v(s) = 0, \tag{2.8}$$

where

$$\begin{aligned}u(s) &= -\beta'(s) + \beta(s)b + \frac{1}{2}\beta(s)^2 + l_1[\mathbb{E}(e^{\beta(s)Z_i}) - 1)] \\ v(s) &= -\alpha'(s) + \beta(s)a + l_0[\mathbb{E}(e^{\beta(s)Z_i}) - 1] + l_1\mathbb{E}[Z_ie^{\beta(s)Z_i}].\end{aligned} \tag{2.9}$$

Because (2.8) must hold for all x, we have $u(s) = v(s) = 0$ for all $s \in \mathbb{R}$. Therefore, we can reduce the PDE to a set of ODE's, namely:

$$\begin{aligned}\beta'(s) &= \beta(s)b + \frac{1}{2}\beta(s)^2 + l_1[\mathbb{E}(e^{\beta(s)Z_i}) - 1)] \\ \alpha'(s) &= \beta(s)a + l_0[\mathbb{E}(e^{\beta(s)Z_i}) - 1] + l_1\mathbb{E}[Z_ie^{\beta(s)Z_i}].\end{aligned} \tag{2.10}$$

Solving this system of ODE's leads to a solution of the PDE and therefore to a solution for (2.3), i.e., for the characteristic function of X.

THEOREM 1: *Let $X = X^x$ be the solution of the stochastic differential equation defined in* (2.1) *with initial condition $X_0 = x$ for all $x \in \mathcal{X}$ for some closed subset $\mathcal{X} \subset \mathbb{R}$. Assume that X is affine as in Definition 1. Further, assume that the jump intensity λ is affine in X. Then the drift and the variance have affine dependence on the current state X_s.*

Beweis. See Appendix A. □

REMARK 1: To simplify the proof of Theorem 1 and as it is enough for our purpose, we assume an affine jump intensity λ. For a more general result, we refer to Duffie et al. (2003), Theorem 2.12.

2.3 A review of Lin and Chang's results

In Lin and Chang's model, the forward price of the S&P 500 index, denoted as F_t^T, is modeled as a jump-diffusion process with stochastic instantaneous variance v_t. Under the risk-neutral measure $\mathbb{Q} \sim \mathbb{P}$, these processes are defined as

$$d \ln F_t^T = -\frac{1}{2} v_t dt + \sqrt{v_t} d\omega_{S,t} + z_S dN_t - \kappa \lambda_t dt, \tag{2.11}$$

$$dv_t = \kappa_v(\theta_v - v_t)dt + \sigma_v \sqrt{v_t} d\omega_{v,t} + z_v dN_t, \tag{2.12}$$

where $\omega_{S,t}$ and $\omega_{v,t}$ are two $\mathbb{Q}$-Brownian motions with correlation coefficient ρ. Asset returns and variance jump at the same time according to the poisson process N_t. The variance jump size z_v is exponentially distributed with mean $\mu_v > 0$, i.e., its probability density is given by $p(z_v) = \frac{1}{\mu_v} e^{-\frac{z_v}{\mu_v}}$, $0 \leq z_v < +\infty$. To introduce correlated jump sizes, the asset return jump size z_S is conditioned on the realization of z_v. In particular, z_S is normally distributed with mean $\mu_j + \rho_j z_v$ and variance σ_j^2. The jump intensity is assumed to be $\lambda_t = \lambda_0 + \lambda_1 v_t$ and the relative forward price jump size, $J \equiv e^{z_S} - 1$, has a mean given by[2]

$$\kappa \equiv \mathbb{E}^{\mathbb{Q}}(e^{z_S} - 1) = \mathbb{E}^{\mathbb{Q}}[\mathbb{E}^{\mathbb{Q}}(e^{z_S}|z_v)] - 1 = \mathbb{E}^{\mathbb{Q}}\left(e^{\mu_j + \rho_j z_v + \frac{1}{2}\sigma_j^2}\right) - 1 = \frac{e^{\mu_j + \frac{1}{2}\sigma_j^2}}{1 - \rho_j \mu_v} - 1.$$

The variance and covariance of the two jump sizes, z_v and z_S, are given by

$$\begin{aligned}
&\mathrm{Var}(z_v) = \mathbb{E}^{\mathbb{Q}}[(z_v - \mu_v)^2] = \mathbb{E}^{\mathbb{Q}}(z_v^2) - \mu_v^2 = \mu_v^2, \\
&\mathrm{Var}(z_S) = \mathbb{E}^{\mathbb{Q}}[(z_S - \mu_j - \rho_j \mu_v)^2] = \mathbb{E}^{\mathbb{Q}}\{[(z_S - \mu_j - \rho_j z_v) + \rho_j(z_v - \mu_v)]^2\} = \sigma_j^2 + \rho_j^2 \mu_v^2, \\
&\mathrm{Cov}(z_S, z_v) = \mathbb{E}^{\mathbb{Q}}[(z_S - \mu_j - \rho_j \mu_v)(z_v - \mu_v)] = \mathbb{E}^{\mathbb{Q}}[\rho_j (z_v - \mu_v)^2] = \rho_j \mu_v^2,
\end{aligned}$$

[2]It seems to us that the notation J_t, frequently used in the literature including Lin and Chang, is not appropriate because J is a random number instead of a process.

hence the correlation coefficient between z_S and z_v is given by

$$\frac{\mathrm{Cov}(z_S, z_v)}{\sqrt{\mathrm{Var}(z_S)\cdot\mathrm{Var}(z_v)}} = \frac{\rho_j\mu_v}{\sqrt{\sigma_j^2+\rho_j^2\mu_v^2}}.$$

The variance process can be rewritten as

$$dv_t = \kappa_v^*(\theta_v^* - v_t)dt + \sigma_v\sqrt{v_t}d\omega_{v,t} + z_v dN_t - (\lambda_0 + \lambda_1 v_t)\mu_v dt,$$

where

$$\kappa_v^* = \kappa_v - \lambda_1\mu_v, \qquad \theta_v^* = \frac{\kappa_v\theta_v + \lambda_0\mu_v}{\kappa_v - \lambda_1\mu_v},$$

are the effective mean-reverting speed and long-term mean level under the risk-neutral measure $\mathbb{Q}$. Note that the mean of jump process, $\mathbb{E}^{\mathbb{Q}}(z_v dN_t) = (\lambda_0 + \lambda_1 v_t)\mu_v dt$, affects the parameter values of the mean-reversion process. Based on the CBOE definition, the VIX squared can be derived from[3]

$$\begin{aligned}
\mathrm{VIX}_t^2 &\equiv \frac{2}{\tau}\mathbb{E}_t^{\mathbb{Q}}\left[-\ln\frac{S_{t+\tau}}{F_t^{t+\tau}}\right] = \frac{2}{\tau}\mathbb{E}_t^{\mathbb{Q}}\left[\int_t^{t+\tau}\frac{dS_t}{S_t} - d(\ln S_t)\right], \\
&= \frac{2}{\tau}\mathbb{E}_t^{\mathbb{Q}}\left[\int_t^{t+\tau}\frac{dF_t^T}{F_t^T} - d(\ln F_t^T)\right], \qquad F_t^T = S_t e^{r(T-t)}, \\
&= \frac{2}{\tau}\mathbb{E}_t^{\mathbb{Q}}\left\{\int_t^{t+\tau}\left[\frac{1}{2}v_t + (e^{z_S} - 1 - z_S)(\lambda_0 + \lambda_1 v_t)\right]dt\right\} \\
&= \frac{\zeta_1}{\tau}\mathbb{E}_t^{\mathbb{Q}}\left(\int_t^{t+\tau} v_t dt\right) + \zeta_2 = \frac{\zeta_1}{\tau}(a_\tau v_t + b_\tau) + \zeta_2, \qquad (2.13)
\end{aligned}$$

where $\tau = 30/365$ and

$$\begin{aligned}
&\zeta_1 = 1 + 2\lambda_1[\kappa - (\mu_j + \rho_j\mu_v)], \qquad \zeta_2 = 2\lambda_0[\kappa - (\mu_j + \rho_j\mu_v)], \\
&a_\tau = \frac{1 - e^{\kappa_v^*\tau}}{\kappa_v^*}, \qquad b_\tau = \theta_v^*(\tau - a_\tau).
\end{aligned}$$

[3]The result here is the same as Lin and Chang's, but the derivation is slightly different from that of Lin and Chang, in which they introduce an approximation on $\ln(1+J)$, which seems to be unnecessary at least in our view.

Denoting $L = \ln S$, the price of a European call option $C(\tau_C, L, v)$ written on VIX with the strike price K and time-to-maturity $\tau_C \equiv T - t$ satisfies the following integro-partial differential equation (IPDE)[4]

$$\begin{aligned}
&\frac{1}{2}v\frac{\partial^2 C}{\partial L^2} + \left[r - \lambda_0\kappa - \left(\lambda_1\kappa + \frac{1}{2}\right)v\right]\frac{\partial C}{\partial L} + \rho\sigma_v v\frac{\partial^2 C}{\partial L\partial v} \\
&\quad + \frac{1}{2}\sigma_v^2 v\frac{\partial^2 C}{\partial v^2} + \kappa_v(\theta_v - v)\frac{\partial C}{\partial v} - \frac{\partial C}{\partial \tau_C} - rC \\
&\quad + \mathbb{E}_t^{\mathbb{Q}}\{[\lambda_0 + \lambda_1(v + z_v)]C(\tau_C, L + z_S, v + z_v) - (\lambda_0 + \lambda_1 v)C(\tau_C, L, v)\} = 0,
\end{aligned}$$

with final condition $C(\tau_C = 0, L, v) = \max(\text{VIX}_T - K, 0)$, where $\text{VIX}_T = \sqrt{\zeta_1 a_\tau v_T/\tau + \zeta_1 b_\tau/\tau + \zeta_2}$.

Lin and Chang claim that they have obtained a closed-form VIX option pricing formula as follows

$$C(\tau_C, L, v) = F_t^{VIX}(T)e^{-r\tau_C}\Pi_1 - Ke^{-r\tau_C}\Pi_2, \tag{2.14}$$

where

$$\Pi_1 = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left[\frac{e^{-i\phi\ln K^2}f_2(\tau_C; i\phi + 1/2)}{i\phi f_2(\tau_C; 1/2)}\right]d\phi, \tag{2.15}$$

$$\Pi_2 = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left[\frac{e^{-i\phi\ln K^2}f_2(\tau_C; i\phi)}{i\phi}\right]d\phi, \tag{2.16}$$

and $f_2(\tau_C; i\phi) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{i\phi\ln \text{VIX}_T^2}\right]$, the characteristic function of $\ln \text{VIX}_T^2$ given by

$$f_2(\tau_C; i\phi) = \exp[C_2(\tau_C) + J_2(\tau_C) + D_2(\tau_C)\ln \text{VIX}_t^2], \tag{2.17}$$

where $D_2(\tau_C)$, $C_2(\tau_C)$ and $J_2(\tau_C)$ are defined in equation (B.10) in Lin and Chang (2010).

[4] In Lin and Chang, both variables t and τ_C are used as independent variables in the option price function, $C(t, \tau_C)$. Here we choose to use one of them, τ_C, as they are related by $\tau_C \equiv T - t$.

2.4 Disproving the correctness of Lin and Chang's formula

2.4.1 Formal Argument

We start by presenting the following result, which is based on a formal argument outlined in the appendix.

PROPOSITION 1: *Lin and Chang's formula (2.14, 2.15, 2.16, 2.17) with $D_2(\tau_C)$, $C_2(\tau_C)$ and $J_2(\tau_C)$ given by their equations (B.10) in Lin and Chang (2010) is not an exact solution of their pricing equation (2.14).*

Beweis. See Appendix B. □

We first note that using equations (2.14, 2.15, 2.16), Lin and Chang describe VIX option price in terms of the characteristic function of $\ln \mathrm{VIX}_T^2$, i.e., $f_2(\tau_C; i\phi)$. This representation is fine because it is consistent with Bakshi and Madan (2000). The key issue here is the analytical tractability of $f_2(\tau_C; i\phi)$, without which the representation does not help us much in computing the VIX option prices.

When Lin and Chang solve the problem, they conjecture in their equation (B.4) that the characteristic function of $\ln \mathrm{VIX}_T^2$ has the following form:

$$f_2(\tau_C; i\phi) \equiv \mathbb{E}_t^{\mathbb{Q}}\left[e^{i\phi \ln \mathrm{VIX}_T^2}\right] = e^{C_2(\tau_C)+J_2(\tau_C)+D_2(\tau_C)\ln \mathrm{VIX}_t^2+G_2(\tau_C)L_t}. \tag{2.18}$$

By imposing such a structure, they implicitly assume that $C_2(\tau_C)$, $J_2(\tau_C)$ and $D_2(\tau_C)$ are not functions of VIX_t when they derive ODEs for them. However, in the final result of their equation (B.10), $C_2(\tau_C)$, $J_2(\tau_C)$, and $D_2(\tau_C)$ are indeed functions of VIX_t, which contradicts their original assumption. Therefore, their conjecture (2.18) cannot be appropriate.

We also note that during the process of solving for $f_2(\tau_C; i\phi)$, Lin and Chang (2010) introduce an approximation in their equation (B.6) for $\exp[i\phi \ln(1 + (\mu_v/\mathrm{VIX}_T^2))]$ by using Taylor's expansion at VIX_t^2. However, the error of this approximation is not analyzed.[5]

What is the reason the method used by Lin and Chang (2010) fails? To give an answer to this question, we observe the following:

[5] Also, the variable M in equations below their equation (B.6) is never defined in Lin and Chang (2010).

PROPOSITION 2: *The characteristic function of the stochastic process* $\ln(\mathrm{VIX}_t^2)$ *cannot be exponentially affine in* $\ln(\mathrm{VIX}_t^2)$.

Beweis. See Appendix C. □

The derivation of Proposition 2 in the appendix makes it obvious why the method used by Lin and Chang (2010) fails, namely because of non-affine dependence of the drift, variance and jump on $\ln(\mathrm{VIX}_t^2)$. A potential remedy to obtain at least a closed-form approximation for the characteristic function would be to apply a second-order perturbation of $\ln(\mathrm{VIX}_t^2)$ around some fixed volatility level. Such an approximation would lead to a characteristic function that is exponential linear-quadratic in VIX_t^2. However, in such a setting, additional care has to be applied to the specification of the volatility dynamics in a setting with jumps (see, e.g., Cheng and Scaillet (2007)).

2.4.2 Numerical Investigation

So far, we have presented a formal argument that Lin and Chang's formula for VIX option pricing cannot be correct. However, one might argue that their formula may produce reasonable prices and may therefore serve as an approximation of the true option pricing formula. Being an approximate formula for the prices of VIX options and futures, its accuracy is important for users. Unfortunately, with some numerical analysis, we find that in general, Lin and Chang's formula (2.14, 2.15, 2.16, 2.17) clearly misprices VIX options and futures. Furthermore, the error could be substantially large.

To substantiate our claim, we use a simplified case to analyze the error of Lin and Chang's formula. In particular, we use the classical Heston model for stochastic volatility (Heston (1993)). Under such a specification, the conditional risk-neutral probability density function of VIX_T, $f^{\mathbb{Q}}(\mathrm{VIX}_T|\mathrm{VIX}_t)$ has been provided by Zhang and Zhu (2006), which can be used to calculate the prices of VIX futures and options given by

$$\mathrm{VIXF}_t^T = \mathbb{E}_t^{\mathbb{Q}}[\mathrm{VIX}_T], \tag{2.19}$$

$$C(T-t, L, v) = e^{-r(T-t)}\mathbb{E}_t^{\mathbb{Q}}[\max(\mathrm{VIX}_T - K, 0)]. \tag{2.20}$$

Using the parameter values estimated from the VIX time series from January 2, 1990 to March 1, 2005 by Zhang and Zhu (2006), $(\kappa_v, \theta_v, \sigma_v) = (4.9179, 0.04874, 0.4868)$, and we assume the current VIX level is at 15% and the riskfree rate is $r = 2\%$. The prices of VIX futures and options with different maturities are presented in Table 2.1 and 2.2.

As we can see from the Tables, Lin and Chang's formula (2.14, 2.15, 2.16, 2.17) misprices VIX options and futures. The error could be substantially large.[6] Lin and Chang formula overprices two-month VIX options by about 40%. The overpricing could be even higher than 100% for one-year VIX options. The overpricing for VIX futures is also large even though it is smaller than that for VIX options.

In equation (B.10) in Lin and Chang (2010), The variable B appears in $e^{B\tau_C}$, therefore $B\tau_C$ has to be dimensionless. However, from the formula for B, we can tell that it is not dimensionless due to the last term $1/\ln \mathrm{VIX}_t^2$. This indicates that the formula for B has some problems. Indeed, note that for VIX futures and options with a very long maturity, i.e., $T - t \to +\infty$, we have

$$\lim_{T-t\to+\infty} v_T = \theta_v^*,$$

and

$$\lim_{T-t\to+\infty} \mathrm{VIX}_T = \sqrt{\zeta_1\theta_v^* + \zeta_2}.$$

Then the VIX futures price has the same limit

$$\lim_{T-t\to+\infty} \mathrm{VIXF}_t^T = \sqrt{\zeta_1\theta_v^* + \zeta_2}, \tag{2.21}$$

and the forward VIX call option price has the limit as follows

$$\lim_{T-t\to+\infty} e^{r(T-t)}C(T-t, L, v) = \max(\sqrt{\zeta_1\theta_v^* + \zeta_2} - K, 0). \tag{2.22}$$

The asymptotic behavior of Lin and Chang's formula, depending on the sign of the value of B, does not follow the property above in general.

[6]Note, the VIX options with a maturity of one to two months are the most liquid ones.

Tabelle 2.1: The prices of VIX futures with different maturities. The parameter values of the Heston (1993) model are taken to be $(\kappa_v, \theta_v, \sigma_v) = (4.9179, 0.04874, 0.4868)$ that are estimated from the VIX time series from January 2, 1990 to March 1, 2005 by Zhang and Zhu (2006). The current VIX level is $\text{VIX}_0 = 15$. LC is obtained by using Lin and Chang's (2010) formula. ZZ is obtained by using Zhang and Zhu (2006) formula. RE is the relative error between LC and ZZ, computed as $LC/ZZ - 1$.

Maturity (year)	LC	ZZ	RE (%)
0.0	15.00	15.00	0.0
0.1	18.13	17.60	3.0
0.2	20.87	19.09	9.3
0.3	23.20	19.95	16.3
0.4	25.16	20.46	23.0
0.5	26.79	20.77	29.0
0.6	28.13	20.96	34.2
0.7	29.23	21.07	38.7
0.8	30.13	21.14	42.5
0.9	30.87	21.18	45.7
1.0	31.47	21.20	48.4
1.1	31.95	21.22	50.5
1.2	32.35	21.23	52.3

Tabelle 2.2: The prices of VIX call options with different maturities. The parameter values of the Heston's Heston (1993) model are taken to be $(\kappa_v, \theta_v, \sigma_v) = (4.9179, 0.04874, 0.4868)$ that are estimated from the VIX time series from January 2, 1990 to March 1, 2005 by Zhang and Zhu (2006). The current VIX level is $\text{VIX}_0 = 15$ and riskfree rate is $r = 2\%$. LC is obtained by using Lin and Chang's (2010) formula. ZZ is obtained by using Zhang and Zhu (2006) approach. RE is the relative error between LC and ZZ, computed as $LC/ZZ - 1$.

Maturity (year)	LC	ZZ	RE (%)
0.0	0.00	0.00	0.0
0.1	4.03	3.17	27.1
0.2	6.69	4.51	48.5
0.3	8.90	5.29	68.4
0.4	10.73	5.75	86.7
0.5	12.23	6.02	103.2
0.6	13.47	6.18	117.9
0.7	14.47	6.27	130.7
0.8	15.28	6.32	141.7
0.9	15.94	6.35	151.0
1.0	16.46	6.36	158.8
1.1	16.88	6.36	165.3
1.2	17.21	6.36	170.0

2.5 Conclusion

In this note, we prove that Lin and Chang's (2009, 2010) formula is not an exact solution of their pricing equation. Using as a reduced specification the simple case of the Heston (1993) model, we demonstrate that Lin and Chang's formula misprices VIX futures and options in general and the error could be substantially large. We further point out that an exact formula has actually been provided by Sepp (2008).

The empirical features on VIX options market provided by Lin and Chang (2010) are based on their in-accurate formula. They need to be reexamined immediately by using the correct VIX option pricing formula. Other research that uses Lin and Chang's formula such as, e.g., Wang and Daigler (2011) and Chung et al. (2011) also needs to be reexamined.

Appendix

Proof of Theorem 1

Define the function

$$M(X_s, s) = e^{\phi(t-s,z)+\psi(t-s,z)X_s}. \tag{2.23}$$

Using Itô's formula as in (2.6) we obtain the equation

$$\begin{aligned} 0 =& M(X_s, s)\big(-\partial_t\phi(t-s,z) - \partial_t\psi(t-s,z)X_s + \psi(t-s,z)\mu(X_s) \\ &+ \frac{1}{2}\psi(t-s,z)^2\sigma(X_s)^2 + \mathbb{E}\left[\lambda(X_s+Z_i)M(Z_i,s) - \lambda(X_s)\right]\big) \end{aligned} \tag{2.24}$$

for all $s \leq t$. Letting $s \to 0$ and dividing by $M(x, 0)$, we thus obtain

$$\begin{aligned} &\partial_t\phi(t,z) + \partial_t\psi(t,z)x \\ &= \psi(t,z)\mu(x) + \frac{1}{2}\psi(t,z)^2\sigma(x)^2 + \lambda_0\mathbb{E}[M(Z_i) - 1] + \lambda_1\mathbb{E}[Z_iM(Z_i,0)] + \lambda_1\mathbb{E}[M(Z_i,0) - 1]x \end{aligned} \tag{2.25}$$

for all $x \in \mathcal{X}$ and $t \geq 0$, where we have written $\lambda(x) := \lambda_0 + \lambda_1 x$. Now since $\psi(0, z) = z$ we see that μ and σ^2 have to be affine in x.

Proof of Proposition 1

Consider the special case of no-jump, i.e., $z_S = z_v = 0$, $\lambda_0 = \lambda_1 = 0$, hence $\kappa \equiv \mathbb{E}(e^{z_S} - 1) = 0$, $\kappa_v^* = \kappa_v$ and $\theta_v^* = \theta_v$. Then, the VIX formula simplifies to

$$\mathrm{VIX}_t^2 = \frac{1}{\tau}(a_\tau v_t + b_\tau),$$

where $a_\tau = \dfrac{1-e^{\kappa_v \tau}}{\kappa_v}$, $b_\tau = \theta_v(\tau - a_\tau)$. Note that $\zeta_1 = 1$ and $\zeta_2 = 0$. The VIX option pricing problem becomes

$$\begin{aligned}
&\frac{1}{2}v\frac{\partial^2 C}{\partial L^2} + \left(r - \frac{1}{2}v\right)\frac{\partial C}{\partial L} + \rho\sigma_v v\frac{\partial^2 C}{\partial L \partial v} \\
&\qquad + \frac{1}{2}\sigma_v^2 v\frac{\partial^2 C}{\partial v^2} + \kappa_v(\theta_v - v)\frac{\partial C}{\partial v} - \frac{\partial C}{\partial \tau_C} - rC = 0, \\
&C(\tau_C = 0, L, v) = \max(\mathrm{VIX}_T - K, 0).
\end{aligned} \tag{2.26}$$

The Lin and Chang's formula becomes

$$C(\tau_C, L, v) = F_t^{VIX}(T)e^{-r\tau_C}\Pi_1 - Ke^{-r\tau_C}\Pi_2, \tag{2.27}$$

where

$$\Pi_1 = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left[\frac{e^{-i\phi\ln K^2} f_2(\tau_C; i\phi + 1/2)}{i\phi f_2(\tau_C; 1/2)}\right] d\phi, \tag{2.28}$$

$$\Pi_2 = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left[\frac{e^{-i\phi\ln K^2} f_2(\tau_C; i\phi)}{i\phi}\right] d\phi, \tag{2.29}$$

and

$$\begin{aligned}
f_2(\tau_C; i\phi) &= \exp[C_2(\tau_C) + D_2(\tau_C)\ln \mathrm{VIX}_t^2], \qquad (2.30)\\
C_2(\tau_C) &= \frac{B}{A}\kappa_v\tau_C - \frac{\kappa_v}{A}\left\{B\tau_C - \ln\left\{\frac{A}{B} + \left[\left(i\phi + \frac{B}{A}\right)^{-1} - \frac{A}{B}\right]e^{B\tau_C}\right\}\right. \\
&\quad \left. + \ln\left[\left(i\phi + \frac{B}{A}\right)^{-1}\right]\right\}, \\
D_2(\tau_C) &= -\frac{B}{A} + \left\{\frac{A}{B} + \left[\left(i\phi + \frac{B}{A}\right)^{-1} - \frac{A}{B}\right]e^{B\tau_C}\right\}^{-1}, \\
A &= \frac{1}{2}\sigma_v^2\left(\frac{\tau\mathrm{VIX}_t^2}{a_\tau} - \frac{b_\tau}{a_\tau}\right)\left(\frac{a_\tau}{\tau\mathrm{VIX}_t^2}\right)^2\left(\frac{1}{\ln\mathrm{VIX}_t^2}\right), \\
B &= \left[\kappa_v\theta_v - \frac{1}{2}\sigma_v^2\frac{a_\tau}{\tau\mathrm{VIX}_t^2}\left(\frac{\tau\mathrm{VIX}_t^2}{a_\tau} - \frac{b_\tau}{a_\tau}\right) + \kappa_v\frac{b_\tau}{a_\tau}\right]\left(\frac{a_\tau}{\tau\mathrm{VIX}_t^2}\right)\left(\frac{1}{\ln\mathrm{VIX}_t^2}\right).
\end{aligned}$$

Because $f_2(\tau_C; i\phi) \equiv \mathbb{E}_t^{\mathbb{Q}}\left[e^{i\phi \ln \mathrm{VIX}_T^2}\right]$ is the characteristic function of $\ln \mathrm{VIX}_T^2$, it must be a solution of pricing PDE (2.26). However, by substituting equation (2.30) into equation (2.26), we can show that it is not a solution of (2.26). Therefore, Lin and Chang's formula (2.14, 2.15, 2.16, 2.17) with $D_2(\tau_C)$, $C_2(\tau_C)$ and $J_2(\tau_C)$ given by their equations (B.10) in Lin and Chang (2010) is not an exact solution of their pricing equation (2.14).

Proof of Proposition 2

Recall the equation

$$\mathrm{VIX}_t^2 = a \cdot \nu_t + b, \tag{2.31}$$

by (8) of Lin and Chang (2010), where $a, b \in \mathbb{R}$ are defined as in Lin and Chang. Equivalently, we can write

$$\ln(\mathrm{VIX}_t^2) = \ln(a \cdot \nu_t + b). \tag{2.32}$$

Using Itô's formula, equation (2.32) transforms to

$$\begin{aligned} d\ln(\mathrm{VIX}_t^2) =& \Big(\frac{a}{\mathrm{VIX}_t^2}\kappa_\nu(\theta_\nu - a^{-1}\mathrm{VIX}_t^2 + b) - \frac{1}{2}\frac{a^2\theta_\nu}{(\mathrm{VIX}_t^2)^2}(a^{-1}\mathrm{VIX}_t^2 - b)\Big)dt \\ &+ \frac{a}{\mathrm{VIX}_t^2}\sigma_\nu(\sqrt{a^{-1}\mathrm{VIX}_t^2 - b})d\omega_{\nu,t} \\ &+ (\ln(\mathrm{VIX}_t^2 + az_\nu + b) - \ln(\mathrm{VIX}_t^2))dN_t. \end{aligned} \tag{2.33}$$

Equation (4.2) shows that the drift, the variance and the jump intensity are not affine in $\ln(\mathrm{VIX}_t^2)$ and therefore, by Theorem 1 and Remark 1, the characteristic function of $\ln(\mathrm{VIX}_t^2)$ cannot be exponential affine in $\ln(\mathrm{VIX}_t^2)$.

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Kapitel 3

A Robust Fundamental Theorem of Asset Pricing with Discrete Martingale Measures

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Abstract: The classical version of the Fundamental Theorem of Asset Pricing requires that zero-sets of the real-world probability measure $\mathbb{P}$ are known. We choose a different route and start from a possibly non-dominated set of probability measures $\mathcal{P}$ representing uncertainty about the zero-sets of the real-world probability measure. Since the concept of equivalent measures becomes meaningless under such a framework, we use the notion of $\mathcal{P}$*-full support*, which is a condition on the support of a martingale measure $\mathbb{Q}$. We derive a version of the Fundamental Theorem of Asset Pricing and find that no-arbitrage, in our context, is equivalent to the existence of a discrete martingale measure.

3.1 Introduction

One of the cornerstones in mathematical finance is the Fundamental Theorem of Asset Pricing (FTAP). The FTAP was first formulated for a finite state space by Harrison and Pliska (1981) and later generalized to measurable spaces and continuous-time processes by Delbaen and Schachermayer (1994). In an infinite state space, the classical framework to derive the FTAP requires to fix a prior probability measure as reference measure. The pricing measure is then taken to be equivalent to the reference measure.

However, the recent discussions on the role of uncertainty during the recent financial crisis seem to call for a reformulation of basic financial theories. One critique aims at an assumption which is inherent in almost all financial models used in practice, namely the fixing of a prior probability measure. This assumption implies that the null-sets of events are known to the modeler. Therefore, with regards to the FTAP, it is natural to ask if one can define arbitrage without fixing a single prior probability measure, but instead allowing for multiple prior probability measures and also obtain a version of the FTAP. This problem has been the focus of recent contributions in the mathematical finance literature.

One stream of literature, e.g., Cherny (2007) and Riedel (2014), formulates the FTAP without any prior probabilities at all, while another stream of literature introduces uncertainty using a set of possibly mutually singular probability measures. To our best knowledge, Beißner (2012) was among the first to give a precise definition of arbitrage under such a setup. The major difficulty arises from the fact that the class of multiple priors does not have to be dominated by a single measure. If there is a dominating measure, then the problem becomes classical, as we can simply work with the null sets of the dominating measure.[1]

We emphasize that the case of non-dominated probability measures is not only theoretically challenging, but also implicitly of high practical relevance. As pointed out, e.g., in Biagini and Cont (2006), practitioners specify a derivative pricing model in terms of a parametric family of martingale measures. The parameters are selected by calibrating them to the observed prices.

[1]In contrast to the mathematical finance literature, the literature on financial economics predominantly addresses Knightian uncertainty Knight (1921) in terms of expected return ambiguity. Such ambiguity results in the formulation of a set of equivalent probability measures. See, among many others, Hansen and Sargent (2001), Chen and Epstein (2002), Epstein and Schneider (2003), Leippold et al. (2008), Ju and Miao (2012), and Ulrich (2013). A notable exception and an extension to non-equivalent probability measures is the recent work by Epstein and Ji (2013).

In this procedure, the 'objective' probability measure does not play a role. Then, already in a simple Black-Scholes model, we may obtain different volatilities $\sigma_1 \neq \sigma_2$, which both fit the data well. Clearly, the resulting pricing measures are mutually singular. Hence, the practitioner faces uncertainty in choosing the correct martingale measure. If there is a whole interval $[\underline{\sigma}, \overline{\sigma}]$ of volatilities that fit the data well, we obtain uncountably many non-equivalent probability measures for which, in general, there does not exist a dominating one.[2]

Inspired by the work of Riedel (2014), who replaced the condition of the equivalence of measures in the classical FTAP by the concept of full-support, we aim at formulating a version of the discrete-time FTAP under a more general setup. In a one-period model, our version of the FTAP holds in the case of possibly non-dominated multiple priors as well as in the no-prior case, i.e., under the absence of any prior assumption. Furthermore, we remark that we only require that initial prices are measurable. Hence, compared to Riedel (2014), we require a less restrictive set of assumptions. To obtain a version of the FTAP in a one-period market, Riedel (2014) has to additionally assume that the underlying space is a Polish space and that the derivatives are continuous with respect to the metric. Another early reference is Deparis and Martini (2004), where a FTAP in discrete time is derived and the support of probability measures plays a central role.

Also related to our work is Cherny (2007) and Bouchard and Nutz (2014). Using a similar definition of arbitrage under no priors as in Riedel (2014), Cherny (2007) derives a discrete-time and continuous-time version of the FTAP with a focus on its geometric characterization. In comparison, our setup is more general and allows us to derive the no-prior FTAP of Riedel (2014) and Cherny (2007) as special cases. For the multi-period market model we follow closely the setup of Bouchard and Nutz (2014) and primarily make use of (Bouchard and Nutz, 2014, Lemma 4.6). In comparison to Bouchard and Nutz (2014), we require the stronger assumption that the price process is continuous. This allows us to prove that the existence of *only one* discrete martingale measure, whose conditional supports satisfy certain conditions, already implies the absence of arbitrage in this setup, and vice versa. In general, as shown in Bouchard and Nutz (2014), for every prior one need to construct a corresponding martingale measure.[3] Assuming

[2] For more on uncertain volatility and the role of non-dominated measures in this context, we refer to Soner et al. (2011).

[3] There are other endeavors such as, e.g., Acciaio et al. (2013), that takes another route and derive a (continuous-time) FTAP in a path-wise sense without fixing priors, whereas Bouchard and Nutz (2014) assume a set of

additionally that the priors satisfy the weak Feller property, we show that the absence of arbitrage is equivalent to the existence of a martingale measure, whose conditional and unconditional supports also satisfy certain conditions. Moreover, by imposing the weak Feller property, one still covers many interesting financial market models.

We remark that our analysis of the FTAP under uncertainty covers different situations that may arise in financial models including uncertainty. For instance, we may consider a single investor who has uncertainty about which model to use, i.e., which prior to choose from a set of priors that may be non-dominated. We may also cover a situation, in which multiple investors have non-equivalent beliefs. Finally, we may describe a situation, in which investors have no priors at all, i.e., when investors do not know at all what to believe. Indeed, we can think of this situation as a special case of a multiple-priors setting.

Our paper is structured as follows. In Section 3.2, we start by introducing the basic definitions, for which we closely follow Bouchard and Nutz (2014). We then derive the FTAP in a one-period model under different assumptions with regards to the measurability of the initial prices. In Section 3.4, we extend our analysis to a multi-period setup. Section 3.5 concludes.

3.2 FTAP in the one-period framework

3.2.1 Basic Notation

We first introduce the basic notation, which holds throughout this and the following section for the one-period framework. Let $(\Omega, \mathcal{F})$ be a measurable space and $\mathcal{P}$ be a convex set of probability measures on $(\Omega, \mathcal{F})$. Moreover, $\mathcal{F}_0 \subseteq \mathcal{F}$ is a sigma-algebra.

Definition 1: *A subset $A \subset \Omega$ is called* $\mathcal{P}$-polar*, if $A \subset A'$ for some $A' \in \mathcal{F}$ satisfying $\mathbb{P}(A') = 0$ for all $\mathbb{P} \in \mathcal{P}$. A property is said to hold* $\mathcal{P}$-quasi surely *($\mathcal{P}$-q.s.), if it holds outside a $\mathcal{P}$-polar set.*

For $i \in \{1, \ldots, d\}$ we let $S_0^i : \Omega \to \mathbb{R}$ be an $\mathcal{F}_0$-measurable map, representing the initial price of an asset and $S_1^i : \Omega \to \mathbb{R}$ be an $\mathcal{F}$-measurable map, acting as the future price of the asset. Further, we set $S_1 = (S_1^1, \ldots, S_1^d)$ and $S_0 = (S_0^1, \ldots,\ S_0^d)$.

probability measures representing uncertainty of the real-world measure and derives a (discrete time) FTAP with respect to these measures.

Definition 2: *An $\mathcal{F}_0$-measurable map $H = (H^1, \ldots, H^d) \colon \Omega \to \mathbb{R}^d$ is called a* trading strategy. *We say there exists* no robust arbitrage *or* NRA holds, *if we have*

$$H \cdot \Delta S \geq 0, \quad \mathcal{P}\text{-}q.s. \quad \Rightarrow \quad H \cdot \Delta S = 0, \quad \mathcal{P}\text{-}q.s., \tag{3.1}$$

for all trading strategies H, where $H \cdot \Delta S$ denotes the scalar product of H and $\Delta S = S_1 - S_0$.

We note that robust arbitrage is defined relative to the set $\mathcal{P}$. Whenever we refer to another set of probability measures, we will state that explicitly in order to avoid confusion.
Observe that for $\mathcal{P} = \{\mathbb{P}\}$, the notion of NRA coincides with the usual definition of no arbitrage.

Definition 3: *Let $X, Y : \Omega \to \mathbb{R}^d$ be $\mathcal{F}$-measurable maps. Then, Y is called a* $\mathcal{P}$-version of X, *if the set $\{X \neq Y\}$ is $\mathcal{P}$-polar.*

Definition 4: *A probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ is called* a martingale measure *if $E_{\mathbb{Q}}[|\Delta S'|] < \infty$ and*

$$E_{\mathbb{Q}}[\Delta S' \mid \mathcal{F}_0] = 0 \quad \mathbb{Q}\text{-}a.s. \tag{3.2}$$

for some $\mathcal{P}$-version $\Delta S'$ of ΔS.

Remark 1: *Generally, a martingale measure $\mathbb{Q}$ can be singular to each $\mathbb{P} \in \mathcal{P}$. Therefore, if $\mathbb{Q}$ is a martingale measure for some $\mathcal{P}$-version of ΔS, then it is not necessarily a martingale measure for all $\mathcal{P}$-versions ΔS.*

Definition 5: *A probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ is called* discrete, *if it is of the form*

$$\mathbb{Q} = \sum_{k=1}^{\infty} \alpha_k \delta_{\omega_k},$$

for a sequence $(\alpha_k)_{k \in \mathbb{N}} \subset [0,1]$ and a sequence $(\omega_k)_{k \in \mathbb{N}} \subset \Omega$. Here, δ_ω denotes the point-measure at $\omega \in \Omega$.

Definition 6: *Let $\mathcal{M}$ be a family of probability measures on a Polish space X. We define the* support *of $\mathcal{M}$ by*

$$\mathrm{supp}\mathcal{M} = \bigcap_{\substack{A\subseteq X \ closed, \\ \forall \mu\in\mathcal{M}:\mu(A^c)=0}} A. \tag{3.3}$$

Let $\mathbb{Q}$ be a probability measure on $(\Omega, \mathcal{F})$. The push-forward $\mathbb{Q}\circ(\Delta S)^{-1}$ of $\mathbb{Q}$ under the map ΔS is defined by $(\mathbb{Q}\circ(\Delta S)^{-1})(A) := \mathbb{Q}((\Delta S)^{-1}(A))$ for $A\in\mathcal{B}(\mathbb{R}^d)$ and it is a probability measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$.

Definition 7: *A probability measure $\mathbb{Q}$ on $(\Omega,\mathcal{F})$ is said to have* $\mathcal{P}$-full forward support *if*

$$\mathrm{supp}_{\Delta S}\mathbb{Q} = \mathrm{supp}_{\Delta S}\mathcal{P} \subseteq \mathbb{R}^d, \tag{3.4}$$

where $\mathrm{supp}_{\Delta S}\mathbb{Q} = \mathrm{supp}(\mathbb{Q}\circ(\Delta S)^{-1})$ *and* $\mathrm{supp}_{\Delta S}\mathcal{P} = \mathrm{supp}\{\mathbb{P}\circ(\Delta S)^{-1} \mid \mathbb{P}\in\mathcal{P}\}$ *are the supports of the corresponding push-forward measures on $\mathbb{R}^d$. If Ω is a Polish space, we say that $\mathbb{Q}$ has* $\mathcal{P}$-full support *if*

$$\mathrm{supp}\mathbb{Q} = \mathrm{supp}\mathcal{P}. \tag{3.5}$$

3.2.2 FTAP and Superhedging under multiple priors with constant initial price

We first derive the FTAP under the assumption of multiple priors with measurable future price and $\mathcal{F}_0$ being trivial, meaning that the initial price is assumed to be constant. Under such a setting, the following version of FTAP holds:

Theorem 1: *The following are equivalent:*

1. NRA *holds.*

2. *There exists a discrete $\mathcal{P}$-full forward support martingale measure $\mathbb{Q}$.*

Beweis. By Lemma A.3 in the Appendix, there exists a discrete $\mathcal{P}$-full forward support measure $\hat{\mathbb{Q}}$. Let h be a trading strategy. We have $\hat{\mathbb{Q}}(h\Delta S\geq 0) = \hat{\mathbb{Q}}(\Delta S\in H^h_{\geq}) = \hat{\mathbb{Q}}((\Delta S)^{-1}(H^h_{\geq})) = 1$

if and only if $\text{supp}_{\Delta S}\hat{\mathbb{Q}} \subseteq H^h_{\geq}$, where $H^h_{\geq} := \{y \in \mathbb{R}^d \mid hy \geq 0\}$ is closed. Using Lemma A.2, we can see that we have $\hat{\mathbb{Q}}(h\Delta S > 0) > 0$ if and only if $\text{supp}_{\Delta S}\hat{\mathbb{Q}} \cap H^h_{>} \neq \emptyset$, where $H^h_{>} := \{y \in \mathbb{R}^d \mid hy > 0\}$ is open. Hence, h is a $\hat{\mathbb{Q}}$-arbitrage if and only if $\text{supp}_{\Delta S}\hat{\mathbb{Q}} \subseteq H^h_{\geq}$ and $\text{supp}_{\Delta S}\hat{\mathbb{Q}} \cap H^h_{>} \neq \emptyset$. For the same reasons, we can see that h is a $\mathcal{P}$-arbitrage if and only if $\text{supp}_{\Delta S}\mathcal{P} \subseteq H^h_{\geq}$ and $\text{supp}_{\Delta S}\mathcal{P} \cap H^h_{>} \neq \emptyset$. Since by construction we have $\text{supp}_{\Delta S}\hat{\mathbb{Q}} = \text{supp}_{\Delta S}\mathcal{P}$, we obtain that h is a robust arbitrage if and only if h is a $\hat{\mathbb{Q}}$-arbitrage.

By the classic[4] FTAP, there is no $\hat{\mathbb{Q}}$-arbitrage if and only if there exists a martingale measure $\mathbb{Q}$, which is equivalent to $\hat{\mathbb{Q}}$. Going backwards notice that the same argument proves " $\Leftarrow$ " of the theorem, meaning that we do not need the assumption of $\mathbb{Q}$ being discrete. Now we show that $\text{supp}_{\Delta S}\mathbb{Q} = \text{supp}_{\Delta S}\hat{\mathbb{Q}}$: assume there exists $x \in \text{supp}_{\Delta S}\mathbb{Q} \setminus \text{supp}_{\Delta S}\hat{\mathbb{Q}}$. Then by Lemma A.1 and the fact that $\text{supp}_{\Delta S}\hat{\mathbb{Q}}$ is closed, there exists $\varepsilon > 0$ with $\mathbb{Q}(\Delta S \in B_\varepsilon(x)) > 0$ and $B_\varepsilon(x) \subset \mathbb{R}^d \setminus \text{supp}_{\Delta S}\hat{\mathbb{Q}}$. The latter implies that $\hat{\mathbb{Q}}(\Delta S \in B_\varepsilon(x)) = 0$. Hence, the measures $\mathbb{Q}$ and $\hat{\mathbb{Q}}$ are not equivalent which contradicts our assumption. Therefore, we have shown $\text{supp}_{\Delta S}\mathbb{Q} \subseteq \text{supp}_{\Delta S}\hat{\mathbb{Q}}$. By the same arguments, it follows that $\text{supp}_{\Delta S}\hat{\mathbb{Q}} \subseteq \text{supp}_{\Delta S}\mathbb{Q}$ and finally equality between these two sets.

The Radon-Nikodym theorem implies that

$$\frac{d\mathbb{Q}}{d\hat{\mathbb{Q}}} = f$$

for some positive, $\mathcal{F}$-measurable function $f : \Omega \to \mathbb{R}$. Hence, $\mathbb{Q}$ is also a discrete measure and thus it is the measure we were looking for. $\square$

Remark 2: *Observe that by construction the discrete measure $\mathbb{Q}$ of the theorem can be chosen to be a $\mathcal{P}$- full forward support measure for any $\mathcal{P}$-version $\Delta S'$ of ΔS.*

The above version of the FTAP serves us as a starting point for further generalizations in the one-period case, and finally in the multi-period setting. Before we do so, we discuss the problem of superhedging. For this we denote by $f : \Omega \to \mathbb{R}$ an $\mathcal{F}$-measurable function. From a financial point of view, we interpret f as a contingent claim. Further, we define the *superhedging price* of f by

[4]By the classic FTAP we mean the FTAP under one prior, i.e. where the real-world measure is assumed to be known. As a reference, see for example Delbaen and Schachermayer (1994).

$$\pi(f) := \inf \Big\{ x \in \mathbb{R} \mid \exists H \in \mathbb{R}^d,\ x + H \cdot \Delta S \geq f\ \mathcal{P}\text{-q.s.}\ \Big\}$$

and use the convention

$$\inf \emptyset = +\infty. \tag{3.6}$$

Notice that for any discrete martingale measure $\mathbb{Q}$ we can define $f \colon \Omega \to \mathbb{R}$ making $E_{\mathbb{Q}}[f]$ arbitrary large but satisfying $0 \geq f$ $\mathcal{P}$-q.s., if $\mathcal{P}$ consists of probability measures which have a continuous probability distribution. Therefore, in order to find a duality result, we need to introduce some continuity assumptions and for this a topology on Ω. Let Ω be a Polish space, S_1 be continuous and f be lower semi-continuous. With S_1 being continuous, Theorem 1 also holds for the backward support, which is shown in Theorem A.1. As Lemma A.4 guarantees that a $\mathcal{P}$-full support measure is also a $\mathcal{P}$-full forward support measure, we directly work with the backward support such that in the duality result the supremum is taken over a smaller set. We denote by $\mathcal{M}$ the set of all discrete $\mathcal{P}$-full support martingale measures $\mathbb{Q}$. Moreover, we fix the contingent claim f and denote by $\mathcal{C}$ be the set of all discrete $\mathcal{P}$-full support measures satisfying $\mathbb{E}_{\mathbb{Q}}[|\Delta S| + |f|] < \infty$.

Lemma 1: *Assuming* NRA*, the relative interior* $\mathrm{ri}\{\mathbb{E}_{\mathbb{Q}}[\Delta S] \mid \mathbb{Q} \in \mathcal{C}\}$ *satisfies:*

$$0 \in \mathrm{ri}\{\mathbb{E}_{\mathbb{Q}}[\Delta S] \mid \mathbb{Q} \in \mathcal{C}\} \subseteq \mathbb{R}^d. \tag{3.7}$$

Beweis. As $\mathrm{supp}\mathcal{P} \subseteq \Omega$ is separable, there exists a discrete $\mathcal{P}$-full support measure $\mathbb{Q}'$ and by (Dellacherie and Meyer, 1982, Theorem VII.57), there exists a probability measure $\mathbb{Q}$ equivalent to $\mathbb{Q}'$, satisfying $\mathbb{E}_{\mathbb{Q}}[|\Delta S| + |f|] < \infty$. Therefore, $\mathcal{C}$ is non-empty. Clearly, $\mathcal{C}$ is also convex and the set

$$C := \{\mathbb{E}_{\mathbb{Q}}[\Delta S] \mid \mathbb{Q} \in \mathcal{C}\} \subseteq \mathbb{R}^d$$

is a non-empty, convex subset of $\mathbb{R}^d$. Next, assume that $0 \notin \mathrm{ri}\, C$. We show now that this implies the existence of an arbitrage $h \in \mathbb{R}^d$ or equivalently, we show that there exists $h \in \mathbb{R}^d$ such that $\mathbb{P}(\Delta S \in H^h_{\geq}) = 1$ for all $\mathbb{P} \in \mathcal{P}$ and $\mathbb{P}(\Delta S \in H^h_{>}) > 0$ for some $\mathbb{P} \in \mathcal{P}$, where $H^h_{\geq} := \{y \in \mathbb{R}^d \mid y \cdot h \geq 0\}$ and $H^h_{>} := \{y \in \mathbb{R}^d \mid y \cdot h > 0\}$.

CLAIM 1: *There exists* $h \in \mathbb{R}^d$ *such that* $\mathbb{P}(\Delta S \in H^h_{\geq}) = 1$, *for all* $\mathbb{P} \in \mathcal{P}$.

Proof of Claim: By the separation theorem (see for example Cheridito (2013)), $0 \notin \mathrm{ri}\, C$ implies the existence of $h \in \mathbb{R}^d$ with $h \cdot y \geq 0$ for all $y \in C$. To achieve a contradiction, assume that there exists $\mathbb{P} \in \mathcal{P}$ such that $\mathbb{P}(\Delta S \in H^h_{\geq}) < 1$ or equivalently $\mathbb{P}(\Delta S \in \mathbb{R}^d \setminus H^h_{\geq}) > 0$. Then, by Lemma A.2, we have $\mathrm{supp}\mathbb{P} \cap (\Delta S)^{-1}(\mathbb{R}^d \setminus H^h_{\geq}) \neq \emptyset$ and therefore there exists $\omega^* \in \mathrm{supp}\mathbb{P}$ such that $\Delta S(\omega^*) \in \mathbb{R}^d \setminus H^h_{\geq}$. Let $A := \{\omega \in \Omega \mid \Delta S(\omega) \in \mathbb{R}^d \setminus H^h_{\geq}\}$ such that $\omega^* \in A$. Choose a sequence $(\omega_k)_{k \in \mathbb{N}} \subseteq \Omega$ such that $\overline{\{\omega_k) \mid k \in \mathbb{N}\}} = \mathrm{supp}\mathcal{P}$. Without loss of generality, we can assume $\omega_1 = \omega^*$, since otherwise we just include ω^* into the sequence. Let $(a_k)_{k \in \mathbb{N}} \subseteq (0,1)$ with $\sum_{k=1}^{\infty} a_k = 1$ and define a probability measure by

$$\mathbb{Q}_0' = \sum_{k=1}^{\infty} a_k \delta_{\omega_k}. \tag{3.8}$$

As before, by (Dellacherie and Meyer, 1982, Theorem VII.57), there exists a probability measure $\mathbb{Q}_0$ which is equivalent to $\mathbb{Q}_0'$ such that $\mathbb{E}_{\mathbb{Q}_0}[|\Delta S| + |f|] < \infty$. By equivalence of measures, we then must have $\mathbb{Q}_0(\Delta S \in (H^h_{\geq})^c) > 0$. For $\varepsilon > 0$, let $Z(\omega) := c(1_A(\omega) + \varepsilon)$ with $c := \frac{1}{\mathbb{E}_{\mathbb{Q}_0}[1_A + \varepsilon]} > 0$ and define a probability measure by

$$\mathbb{Q}_1(F) := \int_{\Omega} 1_F(\omega) Z(\omega) d\mathbb{Q}_0(\omega).$$

Since $Z(\omega) > 0$ for all $\omega \in \Omega$ we have that $\mathrm{supp}\mathbb{Q}_1 = \mathrm{supp}\mathbb{Q}_0 = \mathrm{supp}\mathcal{P}$. Hence, $\mathbb{E}_{\mathbb{Q}_1}[|\Delta S| + |f|] < \infty$ and $\mathbb{Q}_1 \in \mathcal{C}$. Further we have

$$\mathbb{E}_{\mathbb{Q}_1}[h \cdot \Delta S] = c \int_{\Omega} \underbrace{1_A h \cdot \Delta S}_{<0} \, d\mathbb{Q}_0 + c \cdot \varepsilon \int_{\Omega} h \cdot \Delta S d\mathbb{Q}_0 < 0 \tag{3.9}$$

for ε small enough and this is a contradiction to the assumption that $h \cdot y \geq 0$ for all $y \in C$. This proves Claim 1 by linearity of the expectation. $\square$

CLAIM 2: *There exists* $\mathbb{P} \in \mathcal{P}$ *with* $\mathbb{P}(\Delta S \in H^h_{>}) > 0$.

Proof of Claim: Since we assumed that $0 \notin \mathrm{ri}\, C$, by the separation theorem (see for example Cheridito (2013)), there exists $y' \in C$ with $h \cdot y' > 0$. By definition of C there exists $\mathbb{Q} \in \mathcal{C}$ with

$\mathbb{E}_{\mathbb{Q}}[\Delta S] = y'$, meaning $\mathbb{E}_{\mathbb{Q}}[h \cdot \Delta S] = h \cdot y' > 0$. Therefore, $\mathbb{Q}(h \cdot \Delta S > 0) > 0$ or equivalently $\mathbb{Q}(\Delta S \in H^h_>) > 0$ which implies $\mathrm{supp}\mathbb{Q} \cap (\Delta S)^{-1}(H^h_>) \neq \emptyset$ by Lemma A.2. Since $\mathbb{Q}$ is a $\mathcal{P}$-full support measure, we also have $\mathrm{supp}\mathcal{P} \cap (\Delta S)^{-1}(H^h_>) \neq \emptyset$, which is by continuity of ΔS and Lemma A.2 equivalent to the existence of $\mathbb{P} \in \mathcal{P}$ such that $\mathbb{P}(\Delta S \in H^h_>) > 0$. Hence, we proved Claim 2.

□

Now, Claims 1 and 2 imply that there exists a $\mathcal{P}$-arbitrage in the market model, hence the assumption $0 \notin \mathrm{ri}\, C$ led to a contradiction. This concludes the proof of Lemma 1. □

The next lemma is a version of (Bouchard and Nutz, 2014, Lemma 3.5), modified to suffice our purpose.

Lemma 2: *Assume* NRA *holds and that* $\pi(f) = 0$. *There exist discrete* $\mathcal{P}$*-full support measures* $\mathbb{Q}_n$, $n \in \mathbb{N}$ *with*

$$\lim_{n\to\infty} \mathbb{E}_{\mathbb{Q}_n}[\Delta S] = 0 \text{ and } \lim_{n\to\infty} \mathbb{E}_{\mathbb{Q}_n}[f] = 0.$$

Beweis. Define

$$C := \{\mathbb{E}_{\mathbb{Q}}[(\Delta S, f)] \mid \mathbb{Q} \in \mathcal{C}\} \subseteq \mathbb{R}^{d+1}.$$

Then, the claim in the lemma is equivalent to $0 \in \bar{C}$, where $\bar{C}$ is the closure of C. Recall that $\mathcal{C}$ is convex and non-empty. Hence, C is a non-empty convex subset of $\mathbb{R}^{d+1}$. Assume that $0 \notin \bar{C}$. Then, by the strong separation theorem (e.g. see (Cheridito, 2013, Thm 3.3.5)) there exists $\phi = (\phi^1, \dots, \phi^{d+1}) \in \mathbb{R}^{d+1}$ and $\alpha > 0$ such that

$$0 < \alpha = \inf_{\mathbb{Q}\in\mathcal{C}} \mathbb{E}_{\mathbb{Q}}[\phi \cdot (\Delta S, f)], \tag{3.10}$$

and by normalizing ϕ, we can assume that $|\phi| = 1$.

Claim 3: $\alpha \leq \phi \cdot (\Delta S, f)$ $\mathcal{P}$*-q.s.*

Beweis. Assume that $\mathbb{P}(\phi \cdot (\Delta S, f) < \alpha) > 0$ for some $\mathbb{P} \in \mathcal{P}$. Then we obtain $\mathrm{supp}\mathbb{P} \cap (\Delta S, f)^{-1}(\mathbb{R}^{d+1} \setminus H^{\phi,\alpha}_{\geq}) \neq \emptyset$, where $H^{\phi,\alpha}_{\geq} := \{y \in \mathbb{R}^{d+1} \mid y \cdot \phi \geq \alpha\}$. Similar to Claim 1, we set $A := \{\omega \in \Omega \mid (\Delta S(\omega), f(\omega)) \in \mathbb{R}^{d+1} \setminus H^{\phi,\alpha}_{\geq}\} \in \mathcal{F}$ and we can find a sequence $(\omega_k)_{k\in\mathbb{N}} \subseteq \Omega$ such that $\omega_1 \in A$ and the measure $\mathbb{Q}_0' = \sum_{k=1}^{\infty} \alpha_k \delta_{\omega_k}$ is a $\mathcal{P}$-full support measure for some sequence $(\alpha_k)_{k\in\mathbb{N}} \subset (0,1)$. Then define $\beta_k \subset (0,1)$ by

$$\beta_1 = 1 - \sum_{k=2}^{\infty} \beta_k, \quad \text{where for } k \geq 2,$$
$$\beta_k = \frac{2^{-k}(\alpha - \phi \cdot (\Delta S(\omega_1), f(\omega_1)))}{\max\{\phi \cdot (\Delta S(\omega_k), f(\omega_k)), |\Delta S(\omega_k)| + |f(\omega_k)|, \alpha - \phi \cdot (\Delta S(\omega_1), f(\omega_1))\}}.$$

We set $\mathbb{Q}_1 = \sum_{k=1}^{\infty} \beta_k \delta_{\omega_k}$ such that $\mathbb{Q}_1 \in \mathcal{C}$ and $\mathbb{E}_{\mathbb{Q}_1}[\phi \cdot (\Delta S, f)] < \alpha$, thereby contradicting inequality (3.10). This proves Claim 3. □

By Claim 3, we have $\alpha \leq \phi' \Delta S + \phi^{d+1} f$ $\mathcal{P}$-q.s. where $\phi' = (\phi^1, \ldots, \phi^d)$. Assume $\phi^{d+1} < 0$. Then we obtain

$$f \leq |(\phi^{d+1})^{-1}| \phi' \Delta S - |(\phi^{d+1})^{-1}| \alpha \quad \mathcal{P}\text{-q.s.}$$

which implies $\pi(f) \leq -|(\phi^{d+1})^{-1}|\alpha < 0$ and thus contradicts our initial assumption $\pi(f) = 0$. Therefore, we must have $0 \leq \phi^{d+1} \leq 1$. Since $-\alpha/2 < 0 = -\phi^{d+1}\pi(f) = -\pi(\phi^{d+1} f)$, there exists $h' \in \mathbb{R}^d$ such that

$$0 < \alpha - \alpha/2 \leq (\phi' + h')\Delta S \quad \mathcal{P}\text{-q.s.},$$

which contradicts our NRA assumption. □

Lemma 3: *Assume* NRA *and let* $\mathbb{Q} \in \mathcal{C}$. *Then, there exists a* $\mathcal{P}$*-full support martingale measure* $\mathbb{Q}'$ *and* $c > 0$ *independent of* $\mathbb{Q}$ *and* $\mathbb{Q}'$ *such that*

$$|\mathbb{E}_{\mathbb{Q}}[f] - \mathbb{E}_{\mathbb{Q}'}[f]| \leq c(1 + |\mathbb{E}_{\mathbb{Q}}[f]|) \, |\mathbb{E}_{\mathbb{Q}}[\Delta S]| \, ,$$

where $|\cdot|$ *denotes the standard norm on* $\mathbb{R}^d$.

Beweis. The proof follows the same steps as (Bouchard and Nutz, 2014, Lemma 3.6), by replacing the sets Θ and Γ in (Bouchard and Nutz, 2014, Lemma 3.6) with the sets $\mathcal{C}$ and $C = \{\mathbb{E}_{\mathbb{Q}}[\Delta S] \mid \mathbb{Q} \in \mathcal{C}\}$. We apply Lemma 1 and realize that for any constants $\alpha, \beta, \gamma > 0$ and any $\mathcal{P}$-full support measures $\mathbb{Q}, \mathbb{Q}'$, we have $\text{supp}\mathcal{P} = \text{supp}(\frac{\alpha\mathbb{Q} + \beta\mathbb{Q}'}{\gamma})$ with $\frac{\alpha}{\gamma}\mathbb{Q}$ denoting the measure $\mathbb{Q}$ rescaled by α/γ. □

Under multiple priors, we obtain the following superhedging result for a one-period model with continuous future price and constant initial price for a lower semi-continuous contingent claim f:

THEOREM 2: *Assume* NRA *holds. Then,*

$$\pi(f) = \sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}}[f]. \tag{3.11}$$

Furthermore, $\pi(f) > -\infty$ *and* $\pi(f) + H \cdot \Delta S \geq f \quad \mathcal{P} - q.s.$ *for some* $H \in \mathbb{R}^d$.

Beweis. " $\leq$ " follows by the same methods as in (Bouchard and Nutz, 2014, Theorem 3.4), using Theorem A.1 and Lemmas 2 and 3. For the other inequality we need to justify why we may apply (Bouchard and Nutz, 2014, Lemma A.2) for our setting, meaning we need to prove the following:

CLAIM 4: $x + H \cdot \Delta S \geq f$ *$\mathcal{P}$-q.s. implies that this holds also $\mathbb{Q}$-a.s. for every* $\mathbb{Q} \in \mathcal{M}$.

Proof of Claim: Assume in contradiction that there exists $\mathbb{Q} \in \mathcal{M}$ and $\omega_k \in \Omega$ such that $\mathbb{Q}(\omega_k) > 0$ and $x + H \cdot \Delta S(\omega_k) < f(\omega_k)$ holds. Then by continuity of ΔS and lower semi-continuity of f there exists an open set $U \ni \omega_k$ such that this inequality is satisfied on whole U. But as $\mathbb{Q}$ is a $\mathcal{P}$-full support measure, Lemma A.2 implies that there exists $\mathbb{P} \in \mathcal{P}$ such that $\text{supp}\mathbb{P} \cap U \neq \emptyset$ and therefore $\mathbb{P}(U) > 0$, which contradicts the assumption. □

This proves the theorem. □

3.2.3 FTAP with initial σ-algebra generated by a countable partition

The next case which generalizes the previous results is to abandon the assumption of a constant initial price and replace it by the assumption that the initial price is measurable with respect to a σ-algebra which is generated by a countable partition. Therefore, we let $(\Omega, \mathcal{F})$ be a measurable space and $S_1 : \Omega \to \mathbb{R}^d$ an $\mathcal{F}$-measurable map, denoting the price of a risky asset at time $t = 1$. Let $S_0 : \Omega \to \mathbb{R}^d$ be an $\mathcal{F}_0$-measurable map, where $\mathcal{F}_0 \subseteq \mathcal{F}$ is a σ-algebra which is generated by a countable partition $\{\mathfrak{a}_n\}_{n=1,\dots,M} \subseteq \mathcal{F}$, where $M \in \mathbb{N} \cup \{\infty\}$. Let N be the set of all $n \in \{1, \dots, M\}$, such that there exists $\mathbb{P} \in \mathcal{P}$ with $\mathbb{P}(\mathfrak{a}_n) > 0$. For $n \in N$ define $\Omega^n := \mathfrak{a}_n$, $\mathcal{F}^n := \sigma(\mathfrak{a}_n \cap F \mid F \in \mathcal{F})$ and denote by ΔS^n the restriction of $\Delta S := S_1 - S_0$ onto Ω^n and $\mathcal{P}^n := \{\mathbb{P}(\cdot \mid \Omega^n) \mid \mathbb{P} \in \mathcal{P}, \mathbb{P}(\mathfrak{a}_n) > 0\}$, where for $F \in \mathcal{F}$ we define $\mathbb{P}(F \mid \Omega^n) := \frac{\mathbb{P}(F \cap \Omega^n)}{\mathbb{P}(\Omega^n)}$. Note that S_0 is constant on $\mathfrak{a}_n$ since $\mathfrak{a}_n$ is an $\mathcal{F}_0$-atom.

Under multiple priors, we obtain the following FTAP for a one-period model with measurable future price and initial price measurable with respect to a sigma-algebra generated by a countable partition:

Theorem 3: *The following are equivalent:*

1. NRA *holds.*
2. *There exists a discrete $\mathcal{P}$-full forward support martingale measure $\mathbb{Q}$, such that for every $n \in N$ the measure $\mathbb{Q}$ restricted to $\mathfrak{a}_n$ is a $\mathcal{P}^n$-full forward support measure for the market model* $(\Omega^n, \mathcal{F}^n, \Delta S^n)$.

We state the proof of Theorem 3 in the Appendix. Using the fact that the restriction of S_0 to any space $(\Omega^n, \mathcal{F}^n)$ for $n \in N$, is a constant, we can use Theorem 1 to obtain a "local"$\mathcal{P}^n$-full forward support martingale measure $\mathbb{Q}^n$ and then 'patch up' the local measures into one measure $\mathbb{Q}$ on the whole of $\Omega = \bigcup_{n \in \{1,\dots,M\}} \mathfrak{a}_n$. The obtained measure $\mathbb{Q}$ still remains a discrete measure, if we accept the axiom of countable choice.

3.2.4 FTAP with no priors

We continue to work under the one-period setting and derive the FTAP, when we do not have any priors. This discussion helps us to put our results in comparison with Riedel (2014) and Cherny (2007). Let $(\Omega, \mathcal{F})$ and S_0, S_1 be as in the previous section and assume that $\mathcal{P}$ contains the set of all delta measures δ_ω for $\omega \in \Omega$. This corresponds to the case, where we are completely uncertain about the real-world probability measure. Clearly, a vector $h \in \mathbb{R}^d$ is then a robust arbitrage if and only if the following holds:

$$h \cdot (S_1(\omega) - S_0(\omega)) \geq 0, \text{ for all } \omega \in \Omega \text{ and} \tag{3.12}$$

$$h \cdot (S_1(\omega) - S_0(\omega)) > 0, \text{ for some } \omega \in \Omega. \tag{3.13}$$

This also corresponds to the definition of (prior-free) arbitrage given in Riedel (2014).

Under "no priors", we obtain the following FTAP for a one-period model with measurable future price and measurable initial price with respect to a countable generated sigma-algebra $\{\mathfrak{a}_n\}$:

THEOREM 4: *The following are equivalent:*

1. NRA *holds.*
2. *There exists a discrete $\mathcal{P}$-full forward support martingale measure $\mathbb{Q}$ such that* $\mathrm{supp}_{\Delta S}\mathbb{Q}(\cdot \mid \mathfrak{a}_n) = \overline{\{\Delta S(\omega) \mid \omega \in \mathfrak{a}_n\}} \subseteq \mathbb{R}^d$ *for every n with $\mathbb{P}(\mathfrak{a}_n) > 0$ for some $\mathbb{P} \in \mathcal{P}$.*

Beweis. The statement follows directly from Theorem 3. □

In particular, for constant initial price, 2. reduces to $\mathrm{supp}_{\Delta S}\mathbb{Q} = \overline{\{\Delta S(\omega) \mid \omega \in \Omega\}}$.

In order to compare Theorem 4 with the corresponding theorems given in Riedel (2014) and Cherny (2007), we restate their theorems and refer to their paper for a proof:

THEOREM 5: *((Riedel, 2014, Theorem 2.3)) Assume that Ω is a Polish space, $\mathcal{F}$ is the Borel σ-algebra of Ω and S_1 is continuous with $S_1^i \geq 0$ for all $i = 1, \ldots, d$ and $S_0 \in \mathbb{R}^d$. Then there is no robust arbitrage, if and only if there exists a martingale measure which assigns positive probability to every open set in Ω.*

THEOREM 6: *((Cherny, 2007, Theorem 2.4)) Let $(\Omega, \mathcal{F})$ be a measurable space. Let $S_1 : \Omega \to \mathbb{R}^d$ be $\mathcal{F}$-measurable and $S_0 \in \mathbb{R}^d$ be constant. The following are equivalent:*

1. $(\Omega, \mathcal{F}, S_0, S_1)$ *is free of robust arbitrage,*
2. $0 \in \mathrm{ri}(C)$, *where* $C = \mathrm{conv}(\overline{\{\Delta S(\omega) \mid \omega \in \Omega\}})$,
3. *For all $F \in \mathcal{F} \setminus \{\emptyset\}$, there exists a martingale measure $\mathbb{P}$ such that $\mathbb{P}(F) > 0$.*

Below, we give an example of a compact space, in which the formulation of the FTAP following Riedel (2014) is invalid, whereas our formulation still works.

EXAMPLE 1: *Let Ω be an uncountable set equipped with the discrete topology. Denote by $X = \beta\Omega$ the Stone-Cech compactification of Ω. Then, each singleton $\{\omega\}$, $\omega \in \Omega$, is open in X.*

REMARK 3: *In the above example, any measure that charges every open set must have total mass infinity, it cannot even be σ-finite. Hence, for such a space there does not exist a probability measure which assigns positive values to all nonempty open sets, i.e., there does not exist a full support measure as used in Riedel (2014). It shows that, in contrast to our Theorem 4, the characterization of Riedel (2014) cannot be formulated for general spaces.*

Consequently, already the weaker condition (2) of Theorem 4 is necessary and sufficient for the absence of robust arbitrage. Similarly, condition (2) of our Theorem 4 is simpler to verify than property (3) of Theorem 6 as derived in Cherny (2007). As we will see in the next chapter, our formulation also allows us to drop the assumption of the initial price S_0 to be generated by a countable σ-algebra.

3.3 FTAP for Polish spaces with measurable initial prices

In this section, we will restrict ourselves to Polish spaces, but therefore allow the initial prices to be measurable. In this framework, we extend our previous results to the one-period FTAP. We follow the setup of Bouchard and Nutz (2014), which will then allow us to derive the corresponding results in a multi-period setting.

Let Ω, Ω_1 be two Polish spaces and $\mathcal{F}_0$ the universal completion of the Borel sigma-algebra of Ω and $\mathcal{F}_1$ be the universal completion of the Borel sigma-algebra of $\Omega \times \Omega_1$. Let $S_0 : \Omega \to \mathbb{R}^d$ as well as $S_1 : \Omega \times \Omega_1 \to \mathbb{R}^d$ be continuous. Define the map $\Delta S : \Omega \times \Omega_1 \to \mathbb{R}^d$ by $\Delta S(\omega, \omega') = S_1(\omega, \omega') - S_0(\omega)$ for $(\omega, \omega') \in \Omega \times \Omega_1$. Let $\mathcal{P}_0$ be a nonempty convex set of probability measures on Ω and for $\omega \in \Omega$, $\mathcal{P}(\omega)$ be a nonempty convex set of probability measures on Ω_1. Then denote $\mathcal{P}$ by the set of probability measures on $\Omega \times \Omega_1$ such that each $\mathbb{P} \in \mathcal{P}$ can be written as

$$\mathbb{P} = \mu_{\mathbb{P}} \times K_{\mathbb{P}}, \tag{3.14}$$

for some probability measure $\mu_{\mathbb{P}} \in \mathcal{P}_0$ and some universally measurable transition kernel $K_{\mathbb{P}}$ from Ω to Ω_1, where $K_{\mathbb{P}}(\omega, \cdot) \in \mathcal{P}(\omega)$ for all $\omega \in \operatorname{supp}\mathcal{P}_0$.

ASSUMPTION 1: *We assume that*

$$\{(\omega, \mathbb{P}) \in \Omega \times \mathcal{M}_1(\Omega_1) \mid \omega \in \operatorname{supp}\mathcal{P}_0, \mathbb{P} \in \mathcal{P}(\omega)\} \quad \text{is analytic.}$$

Note that this assumption guarantees by the Jankov-von Neumann Theorem (see (Bertsekas and Shreve, 1978, Proposition 7.49)), that $\mathcal{P}$ is nonempty in the sense that there exists a universally measurable transition kernel $K_{\mathbb{P}}$ from $\mathrm{supp}\mathcal{P}_0$ to Ω_1 satisfying $K_{\mathbb{P}}(\omega,\cdot) \in \mathcal{P}(\omega)$ for all $\omega \in \mathrm{supp}\mathcal{P}_0$, which can then be extended to Ω in a measurable way.

In the following, for $\omega \in \Omega$ we say that $\mathrm{NA}(\mathcal{P}(\omega))$ holds if there exists no robust arbitrage (with respect to $\mathcal{P}(\omega)$) in the market model with future price $S_1(\omega,\cdot) : \Omega_1 \to \mathbb{R}^d$ and constant initial price $S_0(\omega) \in \mathbb{R}^d$. Further, we denote by $\mathcal{M}_1(\Omega_1)$ the Polish space of all probability measures on Ω_1, equipped with the topology of weak convergence.

LEMMA 4: *Let* NRA *hold. Then, there exists a discrete martingale measure* $\mathbb{Q}$ *on* $\Omega \times \Omega_1$ *of the form*

$$\mathbb{Q} = \mu \times K, \tag{3.15}$$

which satisfies the following properties:

1. μ *is a probability measure on* Ω *of the form* $\mu = \sum_{k=1}^{\infty} a_k \delta_{\omega_k}$ *with* $\{\omega_k \mid k \in \mathbb{N}\} \subseteq \{\omega \in \Omega \mid \mathrm{NA}(\mathcal{P}(\omega))$ *holds* $\}$,
2. $\mathrm{supp}_{S_0}\mu = \mathrm{supp}_{S_0}\mathcal{P}_0$,
3. K *is a probability transition kernel from* $\mathrm{supp}\mathcal{P}_0$ *to* Ω_1,
4. $K(\omega,\cdot)$ *is a discrete martingale measure* $\mathcal{P}_0$*-quasi surely,*
5. $\mathrm{supp}_{\Delta S(\omega,\cdot)}K(\omega,\cdot) = \mathrm{supp}_{\Delta S(\omega,\cdot)}\mathcal{P}(\omega)$ $\mathcal{P}_0$*-quasi surely.*

Beweis. By (Bouchard and Nutz, 2014, Lemma 4.6), the set $N := \{\omega \in \Omega \mid \mathrm{NA}(\mathcal{P}(\omega)) \text{ fails}\}$ is universally measurable and $\mathcal{P}_0$-polar and thus $\mathrm{supp}_{S_0}\mathcal{P}_0 = \mathrm{supp}_{S_0|_{N^c}}\mathcal{P}_0$. As in Lemma A.3, we can choose a sequence $(\omega_k)_{k\in\mathbb{N}} \subseteq N^c$ such that $\overline{\{S_0(\omega_k) \mid k \in \mathbb{N}\}} = \mathrm{supp}_{S_0}\mathcal{P}_0$. We define

$$\mu := \sum_{k=1}^{\infty}(1/2)^k \delta_{\omega_k} \;\text{ if } |\Omega| = \infty, \quad \mu := \frac{1}{|\Omega|}\sum_{k=1}^{|\Omega|}\delta_{\omega_k} \;\text{ if } |\Omega| < \infty.$$

For constructing K, we first show the following claim:

CLAIM 5: *The map* $\Lambda \colon \mathrm{supp}\mathcal{P}_0 \twoheadrightarrow \Omega_1$ *defined by* $\Lambda(\omega) = \mathrm{supp}\mathcal{P}(\omega)$ *is weakly analytic measurable.*

Proof of Claim: For $V \subseteq \Omega_1$ open, we find

$$\begin{aligned}&\{\omega \in \mathrm{supp}\mathcal{P}_0 : \mathrm{supp}\mathcal{P}(\omega) \cap V \neq \emptyset\} \\ &= \{\omega \in \mathrm{supp}\mathcal{P}_0 : K_{\mathbb{P}}(\omega, V) > 0 \text{ for some } \mathbb{P} \in \mathcal{P}\} \\ &= proj_{\Omega}\{\{(\omega, \mathbb{P}) \mid \omega \in \mathrm{supp}\mathcal{P}_0, \mathbb{P} \in \mathcal{P}(\omega)\} \cap \Omega \times Eval_V^{-1}(0, 1]\}.\end{aligned} \tag{3.16}$$

Because of Assumption 1, the map $Eval_V : \mathcal{M}_1(\Omega_1) \to [0, 1]$ defined by $R \mapsto R(V)$ being semicontinuous and the closedness of analytic sets under countable intersections, we deduce that (3.16) is the continuous image of an analytic set. Consequently, the map Λ is weakly analytic measurable. □

Using this claim, we may apply the Castaing representation (Aliprantis and Border, 1999, Corollary 17.14), which yields a sequence of analytic measurable selectors $f_n : \mathrm{supp}\mathcal{P}_0 \to \Omega_1$ for $n \in \mathbb{N}$, such that for all $\omega \in \mathrm{supp}\mathcal{P}_0$, it holds $\Lambda(\omega) = \overline{\{f_n(\omega)\}_{n\in\mathbb{N}}}$. Now we define the set $X := \{(a_n)_{n\in\mathbb{N}} \in l_1 \mid a_n > 0 \text{ and } \sum_{n\in\mathbb{N}} a_n = 1\}$, which is a separable metrizable subspace of l_1. Furthermore, we define $f\colon \mathrm{supp}\mathcal{P}_0 \times X \to \mathbb{R}^d$ by $f(\omega, (a_l)_{l\in\mathbb{N}}) = \sum_{l\in\mathbb{N}} a_l S_1(\omega, f_l(\omega)) - S_0(\omega)$, which is clearly continuous in X and measurable in $\mathrm{supp}\mathcal{P}_0$, showing that f is a Carathéodory function. Finally we define $\Gamma\colon \mathrm{supp}\mathcal{P}_0 \twoheadrightarrow X$ with $\Gamma(\omega) = \{(a_l)_{l\in\mathbb{N}} \in X \mid f(\omega, (a_l)_{l\in\mathbb{N}}) = 0\}$. By (Aliprantis and Border, 1999, Corollary 17.8), Γ has an analytic graph.
To use the Jankov-von Neumann Theorem ((Aliprantis and Border, 1999, Theorem 17.21)), we show that Γ takes non-empty and closed values on a set with $\mathcal{P}_0$-polar complement. Closedness is given as $\Gamma(\omega) = (f(\omega, \cdot))^{-1}(\{0\})$ and f is continuous in X.

Claim 6: *$\Gamma(\omega) \neq \emptyset$ for all $\omega \in N^c$.*

Proof of Claim: Fix $\omega \in N^c$. By Theorem 1, there exists a discrete $\mathcal{P}(\omega)$-full forward support martingale measure $\mathbb{Q}$, which implies that $\mathrm{NA}(\mu_{\mathbb{Q}})$ holds on $\mathbb{R}^d$ by the classic FTAP, where $\mu_{\mathbb{Q}} = \mathbb{Q} \circ (\Delta S)^{-1}$. By (Föllmer and Schied, 2011, Corollary 1.50) this is equivalent to $0 \in \mathrm{ri}(\mathrm{conv}(\mathrm{supp}\mu_{\mathbb{Q}}))$. As $\mathrm{supp}\mathcal{P}(\omega) = \overline{\{f_n(\omega)\}_{n\in\mathbb{N}}}$, we deduce from Lemma A.5,

$$\mathrm{supp}\mu_{\mathbb{Q}} = \mathrm{supp}_{\Delta S(\omega,\cdot)}\mathcal{P}(\omega) = \overline{\{\Delta S(\omega, f_n(\omega))\}_{n\in\mathbb{N}}},$$

where we used the continuity of ΔS. So we obtain $0 \in \mathrm{ri}(\mathrm{conv}(\mathrm{supp}\tilde{\mu}))$ where

$$\tilde{\mu} = \sum_{k=1}^{\infty} \alpha_k \delta_{\Delta S(\omega, f_k(\omega))}$$

for some $\alpha_k \in (0,1)$, $k \in \mathbb{N}$. But this in turn is equivalent to the existence of a martingale measure $\mu' \approx \tilde{\mu}$, hence of the form $\mu' = \sum_{k=1}^{\infty} \beta_k \delta_{\Delta S(\omega, f_k(\omega))}$, with $\beta_k \in (0,1)$ for all $k \in \mathbb{N}$, satisfying $0 = \int_{\mathbb{R}^d} x d\mu'(x) = \sum \beta_k \Delta S(\omega, f_k(\omega))$. Consequently we conclude that $(\beta_k)_{k\in\mathbb{N}} \in \Gamma(\omega)$. $\square$

Hence we may apply the Jankov-von Neumann Theorem giving us a universally measurable selector φ of Γ on N^c, which we extend in a measurable way on $\mathrm{supp}\mathcal{P}_0$. Finally, for A in the universal completion of $\mathcal{B}(\Omega_1)$, we set

$$K(\omega, A) = \sum_{i=1}^{\infty} (\pi_i \circ \varphi(\omega)) \cdot (\mathbb{1}_A(f_i(\omega)))$$

where $\pi_i \colon l_1 \to \mathbb{R}$ is the measurable projection $(a_n)_{n\in\mathbb{N}} \mapsto a_i$. As φ as well as the characteristic function are measurable, K is a universally measurable transition kernel which satisfies by construction properties (4) and, by Lemma A.4, (5) on N^c. By (Bouchard and Nutz, 2014, Theorem 4.5), we know that NRA is equivalent to N being $\mathcal{P}_0$-polar. Setting $\mathbb{Q} = \mu \times K$ we notice that we obtain a discrete measure.

It remains to show that $\mathbb{Q}$ satisfies the martingale property: For $A \in \mathcal{F}_0$ we obtain

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}[\Delta S 1_A] &= \int_\Omega \int_{\Omega_1} 1_A(\omega) \Delta S(\omega, \omega') K(\omega, d\omega') \mu(d\omega) \\
&= \int_\Omega 1_A(\omega) \int_{\Omega_1} \Delta S(\omega, \omega') K(\omega, d\omega') \mu(d\omega) \\
&= \int_{\{\omega_k | k\in\mathbb{N}\}} 1_A(\omega) \underbrace{\int_{\Omega_1} \Delta S(\omega, \omega') K(\omega, d\omega')}_{=0} \mu(d\omega) \\
&\quad + \underbrace{\int_{\{\omega_k | k\in\mathbb{N}\}^c} 1_A(\omega) \int_{\Omega_1} \Delta S(\omega, \omega') K(\omega, d\omega') \mu(d\omega)}_{=0, \text{ since } \mu(\{\omega_k | k\in\mathbb{N}\}^c)=0} = 0,
\end{aligned} \tag{3.17}$$

where in the last equality we have used that $\mu = \sum_{k=1}^{\infty}(1/2)^k\delta_{\omega_k}$ and

$$\int_{\Omega_1} \Delta S(\omega_k, \omega')K(\omega_k, d\omega') = 0$$

for all $k \in \mathbb{N}$ and that the set $\{\omega_k \mid k \in \mathbb{N}\}$ is $\mathcal{B}(\Omega)$-measurable (since points are Borel measurable in Polish spaces) and has μ-measure 1. This shows that $\mathbb{Q}$ is a martingale measure. □

THEOREM 7: *In a one-period model with multiple priors and measurable future and initial price, the following are equivalent:*

1. NRA *holds,*
2. *There exists a discrete martingale measure* $\mathbb{Q}$ *such that*
   (a) $\mathrm{supp}_{S_0}\mathbb{Q}|_\Omega = \mathrm{supp}_{S_0}\mathcal{P}_0$.
   (b) $\mathbb{Q}(\omega, \cdot)$ *is a discrete martingale measure* $\mathcal{P}_0$*-quasi surely,*
   (c) $\mathrm{supp}_{\Delta S(\omega,\cdot)}\mathbb{Q}(\omega, \cdot) = \mathrm{supp}_{\Delta S(\omega,\cdot)}\mathcal{P}(\omega)$ $\mathcal{P}_0$*-quasi surely.*

Beweis. (1) ⇒ (2): This is Lemma 4.
(2) ⇒ (1): Let $N := \{\omega \in \Omega \mid \mathrm{NA}(\mathcal{P}(\omega)) \text{ fails }\}$. By (Bouchard and Nutz, 2014, Lemma 4.6), we know that N is universally measurable. Let

$$A := \{\omega \in \Omega \mid \mathbb{Q}(\omega, \cdot) \text{ is a discrete martingale measure}\},$$
$$B := \{\omega \in \Omega \mid \mathrm{supp}_{\Delta S(\omega,\cdot)}\mathbb{Q}(\omega, \cdot) = \mathrm{supp}_{\Delta S(\omega,\cdot)}\mathcal{P}(\omega)\},$$

such that $A \cap B$ has $\mathcal{P}_0$-polar complement by assumption. By Theorem 1 we notice that $A \cap B \subseteq N^c$ and hence $N \subseteq (A \cap B)^c$ is $\mathcal{P}_0$-polar. By (Bouchard and Nutz, 2014, Theorem 4.5) we conclude that NRA holds. □

DEFINITION 8 (Weak Feller property): *Let K be a probability transition kernel from Ω to Ω_1. We say that K satisfies the* weak Feller property*, if the map $Kf : \Omega \to \mathbb{R}$ defined by $Kf(\omega) := \int_{\Omega_1} f(\omega')K(\omega, d\omega')$ is continuous (in ω) for every $f \in C_b(\Omega_1)$.*

For the next theorem, we make the following assumption.

ASSUMPTION 2: *We assume that $K_{\mathbb{P}}$ satisfies the Feller property for all $\mathbb{P} \in \mathcal{P}$.*

THEOREM 8: *In a one-period model with multiple priors and measurable future and initial price, the following are equivalent:*

1. NRA *holds,*
2. *There exists a discrete martingale measure $\mathbb{Q}$ with $\mathcal{P}$-full forward support such that*
   (a) *$\mathbb{Q}(\omega, \cdot)$ is a martingale measure $\mathcal{P}_0$-quasi surely,*
   (b) *$\mathrm{supp}_{\Delta S(\omega,\cdot)}\mathbb{Q}(\omega, \cdot) = \mathrm{supp}_{\Delta S(\omega,\cdot)}\mathcal{P}(\omega)$ $\mathcal{P}_0$-quasi surely.*

Beweis. (1) $\Rightarrow$ (2): We only have to show that the martingale measure $\mathbb{Q}$ given by Lemma 4 satisfies the $\mathcal{P}$-full forward support property. The continuity of ΔS and Lemma A.4 imply that it is enough to show $\mathrm{supp}\mathbb{Q} = \mathrm{supp}\mathcal{P}$. We first prove the following claim:

CLAIM 7: *Let X be a metric space and $A, B \subseteq X$. If for any open set $O \subseteq X$, $O \cap A \neq \emptyset$ implies $O \cap B \neq \emptyset$, then $\overline{A} \subseteq \overline{B}$.*

Proof of Claim: Let $x \in \overline{A} \setminus \overline{B}$, then $x \in \overline{A} \cap \overline{B}^c$, hence $A \cap \overline{B}^c \neq \emptyset$, but $B \cap \overline{B}^c = \emptyset$. □

Moreover, as every open set $O \subseteq \Omega \times \Omega_1$ is the union of open sets of the form $U \times V$, where $U \subseteq \Omega$ and $V \subseteq \Omega_1$ are open, it is sufficient to prove that

$$\forall\, U \subseteq \Omega \text{ open},\ V \subseteq \Omega_1 \text{ open} : U \times V \cap \mathrm{supp}\mathcal{P} \neq \emptyset \Leftrightarrow U \times V \cap \mathrm{supp}\mathbb{Q} \neq \emptyset. \tag{3.18}$$

Now we fix U and V open and find with Lemma A.2,

$$\begin{aligned} U \times V \cap \mathrm{supp}\mathcal{P} \neq \emptyset &\Leftrightarrow \exists\, \mathbb{P} \in \mathcal{P} : U \times V \cap \mathrm{supp}\mathbb{P} \neq \emptyset \\ &\Leftrightarrow \exists\, \mathbb{P} \in \mathcal{P} : \mu_{\mathbb{P}} \times K_{\mathbb{P}}(U \times V) > 0 \\ &\Leftrightarrow \exists\, \mathbb{P} \in \mathcal{P} : \mu_{\mathbb{P}}\left(\{\omega \in U \mid K_{\mathbb{P}}(\omega, V) > 0\}\right) > 0 \end{aligned} \tag{3.19}$$

For the following, we need a claim, which follows from the Feller property:

CLAIM 8: *Let $\mathbb{P} \in \mathcal{P}$, $\tilde{\omega} \in \Omega$ and $V \subseteq \Omega_1$ open such that $K_{\mathbb{P}}(\tilde{\omega}, V) > 0$.*
Then there exists $\varepsilon > 0$ such that $K_{\mathbb{P}}(\omega, V) > 0$ for all $\omega \in B_\varepsilon(\tilde{\omega})$.

Proof of Claim: By assumption and Lemma A.2 we know that there exists $\omega^\star \in V \cap \operatorname{supp} K_{\mathbb{P}}(\tilde{\omega}, \cdot)$. By Urysohn's Lemma, there exists a bounded continuous function $f : \Omega_1 \to [0, \infty)$ with $f(\omega^\star) = 1$ and $f = 0$ on V^c. Thus $\int_{\Omega_1} f K_{\mathbb{P}}(\tilde{\omega}, d\omega') > 0$. Using the continuity of the map $\omega \mapsto \int_{\Omega_1} f K_{\mathbb{P}}(\omega, d\omega')$, we can see that this map must be positive on an open ball $B_\varepsilon(\tilde{\omega})$. On this ball, $K_{\mathbb{P}}(\omega, V) > 0$ must hold. □

Using this claim, we obtain the following result:

Claim 9: *The following statements are equivalent:*

1. $\exists\, \mathbb{P} \in \mathcal{P} : \mu_{\mathbb{P}} \left(\{\omega \in U \mid K_{\mathbb{P}}(\omega, V) > 0\} \right) > 0$,

2. $\exists\, \mathbb{P} \in \mathcal{P}, \varepsilon > 0, \omega_0 \in U :$
   $B_\varepsilon(\omega_0) \subseteq \{\omega \in U \mid K_{\mathbb{P}}(\omega, V) > 0\}$ *and* $\mu_{\mathbb{P}}(B_\varepsilon(\omega_0)) > 0$,

3. $\exists\, \tilde{\omega} \in \{\omega_k\}_{k \in \mathbb{N}} \cap U : K(\tilde{\omega}, V) > 0$,

where $(\omega_k)_{k \in \mathbb{N}}$ *is a sequence chosen as in Lemma 4.*

Proof of Claim: $(2) \Rightarrow (1)$: This is clear as $B_\varepsilon(\omega_0) \subseteq \{\omega \in U \mid K_{\mathbb{P}}(\omega, V) > 0\}$.
$(1) \Rightarrow (2)$: Because of Lemma A.2, we can choose $\omega_0 \in \{\omega \in U \mid K_{\mathbb{P}}(\omega, V) > 0\} \cap \operatorname{supp} \mu_{\mathbb{P}}$. Then Claim 8 gives an $\tilde{\varepsilon} > 0$ such that $K_{\mathbb{P}}(\omega, V) > 0$ for all $\omega \in B_{\tilde{\varepsilon}}(\omega_0)$. Finally we choose $\varepsilon > 0$ such that $B_\varepsilon(\omega_0) \subseteq B_{\tilde{\varepsilon}}(\omega_0) \cap U$.
$(2) \Rightarrow (3)$:

$$\operatorname{supp}\mu \supseteq \operatorname{supp}\mu_{\mathbb{P}} \Rightarrow \exists \tilde{\omega} \in \{\omega_k\}_{k \in \mathbb{N}} \cap B_\varepsilon(\omega_0)$$
$$\operatorname{supp} K(\tilde{\omega}, \cdot) \supseteq \operatorname{supp} K_{\mathbb{P}}(\tilde{\omega}, \cdot) \Rightarrow K(\tilde{\omega}, V) > 0.$$

$(3) \Rightarrow (2)$: For $\omega_0 = \tilde{\omega}$, it holds that $\operatorname{supp} K(\omega_0, \cdot) = \operatorname{supp} \mathcal{P}(\omega_0)$ and therefore $\operatorname{supp} \mathcal{P}(\omega_0) \cap V \neq \emptyset$. Hence, by Lemma A.2, there exists $K_{\mathbb{P}}(\omega_0, \cdot) \in \mathcal{P}(\omega_0)$ such that $K_{\mathbb{P}}(\omega_0, V) > 0$. Then Claim 8 gives an $\tilde{\varepsilon} > 0$ such that $K_{\mathbb{P}}(\omega, V) > 0$ for all $\omega \in B_{\tilde{\varepsilon}}(\omega_0)$. Now we choose $\varepsilon > 0$ such that $B_\varepsilon(\omega_0) \subseteq B_{\tilde{\varepsilon}}(\omega_0) \cap U$ and $\mu_{\mathbb{P}} \in \mathcal{P}_0$ such that $\operatorname{supp} \mu_{\mathbb{P}} \cap B_\epsilon(\omega_0) \neq \emptyset$, meaning $\mu_{\mathbb{P}}(B_\varepsilon(\omega_0)) > 0$. Finally we set $\mathbb{P} = \mu_{\mathbb{P}} \times K_{\mathbb{P}}$. □

Finally, (3.19) and Claim 9 shows (3.18).

(2) $\Rightarrow$ (1): This follows from Theorem 8. $\square$

The following example shows that if the initial price S_0 is not constant, then the existence of a martingale measure with $\mathcal{P}$-full forward support is not sufficient to imply NRA :

EXAMPLE 2: *Let $\Omega = \Omega_1 = \mathbb{R}$ and for any $(x, y) \in \mathbb{R}^2$ let $S_0(x, y) = x$ and $S_1(x, y) = y$ such that $\Delta S(x, y) = y - x$. Assume $\mathcal{P}$ has just one element $\mathbb{P}$ of the form*

$$\mathbb{P} = p_+^0 \delta_{(0,1)} + p_-^0 \delta_{(0,0)} + \sum_{j \geq 1} p_+^j \delta_{(1/j, 1+1/j)} + p_-^j \delta_{(1/j,0)} \tag{3.20}$$

with $p_+^j, p_-^j > 0$ for all $j \in \mathbb{N} \cup \{0\}$. Then, any $\mathcal{F}_0$-measurable map $h : \mathbb{R} \to \mathbb{R}$, satisfying $h(0) > 0$ and $h(1/j) = 0$ for all $j \in \mathbb{N}$, is a $\mathcal{P}$-arbitrage but the weights $q_\pm^j$ can be chosen such that

$$\mathbb{Q} = \sum_{j \geq 1} q_+^j \delta_{(1/j, 1+1/j)} + q_-^j \delta_{(1/j,0)} \tag{3.21}$$

is a martingale measure and has $\mathcal{P}$-full forward support.

REMARK 4: *The support of a measure on $\Omega \times \Omega_1$ is well-defined, since $\Omega \times \Omega_1$ is a Polish space. Therefore, we could try working with the support on $\Omega \times \Omega_1$ instead of the forward support. In fact, in the proofs of this section, we always work with the support of a measure (as a subset of $\Omega \times \Omega_1$) instead of the forward support, and then conclude with Lemma A.4, by using the continuity of ΔS, that the measure under consideration has also $\mathcal{P}$-full forward support (or conditional full forward support). One could hope that the continuity assumption is redundant if we directly work with the support on $\Omega \times \Omega_1$. However, the following exemplifies that continuity is necessary. It shows that without continuity, even in the case of a constant initial price, the existence of a martingale measure $\mathbb{Q}$ which satisfies* $\mathrm{supp}\mathbb{Q} = \mathrm{supp}\mathcal{P}$ *is not sufficient to imply* NRA. *For this reason, it is more natural to work with the forward support, as this is also defined when $\Omega \times \Omega_1$ is only a measurable space, i.e. is not equipped with a topology.*

EXAMPLE 3: *Let $\Omega = [0, 1]$ be endowed with the usual topology. Let $S_0 = 0$ and $S_1(\omega) = 0$ if $\omega \in \mathbb{Q} \cap (0, 1]$ and 1 otherwise, where $\mathbb{Q}$ denotes the rational numbers. Assume that $\mathcal{P} = \{\mathbb{P}\}$,* *with* $\mathbb{P} = \sum_{\omega_k \in \mathbb{Q}\cap[0,1]} \alpha_k \delta_{\omega_k}$ *for some sequence* $(\alpha_k)_{k\in\mathbb{N}} \subset (0,1)$. *Let* $\mathbb{P}' = \sum_{\omega_k \in \mathbb{Q}\cap(0,1]} \beta_k \delta_{\omega_k}$ *for some sequence* $(\beta_k)_{k\in\mathbb{N}} \subseteq (0,1)$. *Then,* $\mathrm{supp}\mathbb{P} = \mathrm{supp}\mathbb{P}' = [0,1]$ *but* $\mathrm{supp}_{\Delta S}\mathbb{P} = \{0,1\}$ *and* $\mathrm{supp}_{\Delta S}\mathbb{P}' = \{0\}$. *Clearly, the strategy $h = 1$ is a $\mathbb{P}$-arbitrage, while $\mathbb{P}'$ is a martingale measure with* $\mathrm{supp}\mathbb{P}' = \mathrm{supp}\mathbb{P}$.

3.4 Multi-period FTAP on Polish spaces

Let Ω_1 be a Polish space and $T \in \mathbb{N}$ and let $\Omega_t := \Omega_1^t$, $t \in \{0,1,\dots,T\} =: \mathcal{T}$, be the t-fold Cartesian product of Ω_1 with the convention that Ω_0 is a singleton. By $\mathcal{F}_t$ we denote the universal completion of the Borel sigma-algebra on Ω_t. We often interpret $(\Omega_t, \mathcal{F}_t)$ as a subspace of $(\Omega_T, \mathcal{F}_T) =: (\Omega, \mathcal{F})$. Let $S_t \colon \Omega_t \to \mathbb{R}^d$ be continuous for all $t \in \mathcal{T}$ and for $\omega_t \in \Omega_t$ the map $\Delta S_{t+1}(\omega_t, \cdot) \colon \Omega_1 \to \mathbb{R}^d$ defined by $\Delta S_{t+1}(\omega_t, \omega_1) = S_{t+1}(\omega_t, \omega_1) - S_t(\omega_t)$ for all $t \in \mathcal{T} \setminus \{T\}$. Moreover, for $\omega_t \in \Omega_t$ let $\mathcal{P}_t(\omega_t)$ be a nonempty convex set of probability measures on Ω_1 for every $t \in \mathcal{T} \setminus \{T\}$.

Finally we introduce the set $\mathcal{P}$ of probability measures on $(\Omega, \mathcal{F})$ such that each $\mathbb{P} \in \mathcal{P}$ is of the form

$$\mathbb{P} = K_0^{\mathbb{P}}(\omega_0, d\omega_1) \otimes K_1^{\mathbb{P}}(\omega_1, d\omega_2) \otimes \cdots \otimes K_{T-1}^{\mathbb{P}}(\omega_{T-1}, d\omega_T), \tag{3.22}$$

where for all $t \in \mathcal{T} \setminus \{T\}$, $K_t^{\mathbb{P}}$ is a $\mathcal{F}_t$-measurable probability transition kernel from Ω_t to Ω_1 such that $K_t^{\mathbb{P}}(\omega_t, \cdot) \in \mathcal{P}_t(\omega_t)$.

Definition 9: *For $i = 1,\dots,d$ and $t \in \mathcal{T} \setminus \{0\}$ let $H_t^i : \Omega \to \mathbb{R}$ be a $\mathcal{F}_{t-1}$-measurable map. We call $H = (H_t)_{t=1,\dots,T} = (H_t^1,\dots,H_t^d)_{t=1,\dots,T}$* a trading strategy. *The set of all trading strategies is denoted by $\mathcal{H}$. The* gains process *of H is*

$$H \cdot S = (H \cdot S_t)_{t\in\mathcal{T}},\ H \cdot S_t = \sum_{k=1}^{t} H_k \cdot \Delta S_k,\ \text{for } t \in \mathcal{T} \setminus \{0\}. \tag{3.23}$$

Definition 10: *We say there exists* no $\mathcal{P}$-arbitrage *or* NRA holds, *if*

$$H \cdot S_T \geq 0, \quad \mathcal{P}\text{-}q.s. \quad \Rightarrow \quad H \cdot S_T = 0, \quad \mathcal{P}\text{-}q.s., \tag{3.24}$$

for all $H \in \mathcal{H}$.

Definition 11: *A probability measure* $\mathbb{Q}$ *on* $(\Omega, \mathcal{F})$ *is called* a martingale measure *if for all* $t \in \mathcal{T} \setminus \{0\}$ *we have* $E_{\mathbb{Q}}[|\Delta S'_t|] < \infty$ *and*

$$E_{\mathbb{Q}}[\Delta S'_t \mid \mathcal{F}_{t-1}] = 0 \quad \mathbb{Q}\text{-}a.s. \tag{3.25}$$

for some $\mathcal{P}$*-version* $\Delta S'_t$ *of* ΔS_t.

Assumption 3: *We assume that for each* $t \in \mathcal{T} \setminus \{T\}$,

$$\{(\omega, \mathbb{P}) \in \Omega_t \times \mathcal{M}_1(\Omega_1) \mid \omega \in A_t, \mathbb{P} \in \mathcal{P}_t(\omega)\} \quad \text{is analytic, where}$$

$A_t = \{\omega = (\omega_0, \ldots, \omega_t) \mid \omega \in \{\omega_0\} \times \mathrm{supp}\mathcal{P}_0(\omega_0) \times \ldots \times \mathrm{supp}\mathcal{P}_{t-1}(\omega_0, \ldots, \omega_{t-1})\}$.

Note that this assumption guarantees that $\mathcal{P}$ is nonempty in the sense that there exists a universally measurable transition kernel $K_{\mathbb{P}}$ from A_t to Ω_1 satisfying $K_{\mathbb{P}}(\omega, \cdot) \in \mathcal{P}_t(\omega)$ for all $\omega \in A_t$, which can then be extended to Ω_t in a measurable way, as A_t is a closed subset. We finally define a nonempty convex set of probability measures on Ω_t for every $t \in \mathcal{T}$:

$$\tilde{\mathcal{P}}_0 = \{\delta_{\omega_0}\}, \text{ where } \omega_0 \text{ is the single element of } \Omega_0; \quad \text{and for } t \in \mathcal{T} \setminus \{0\},$$
$$\tilde{\mathcal{P}}_t = \{\mathbb{P}_0 \otimes \ldots \otimes \mathbb{P}_{t-1} \mid \mathbb{P}_s \text{ is } \mathcal{F}_s - \text{measurable selector of } \mathcal{P}_s, s = 0, \ldots, t-1\}.$$

Under multiple priors, we obtain the following FTAP for a multi-period model:

Theorem 9: *The following are equivalent:*

(1) NRA *holds.*

(2) There exists a discrete martingale measure $\mathbb{Q}$ *such that for all* $t \in \mathcal{T} \setminus \{T\}$,

  (a) $\mathbb{Q}_t(\omega, \cdot)$ *is a discrete martingale measure* $\tilde{\mathcal{P}}_t$*-quasi surely,*

  (b) $\mathrm{supp}_{\Delta S_{t+1}(\omega, \cdot)} \mathbb{Q}_t(\omega, \cdot) = \mathrm{supp}_{\Delta S_{t+1}(\omega, \cdot)} \mathcal{P}_t(\omega)$ $\tilde{\mathcal{P}}_t$*-quasi surely.*

Beweis. (1)$\Rightarrow$(2): Assume NRA holds. By (Bouchard and Nutz, 2014, Lemma 4.6) we know that the market $(\Omega_t \times \Omega_1, \{\mathcal{F}_t, \mathcal{F}_{t+1}\}, S_t, S_{t+1})$ is free of robust arbitrage for every $t \in \mathcal{T} \setminus \{T\}$. Hence we my define $\mathbb{Q}_0(\omega_0, \cdot)$ as μ is defined in Lemma 4. Then we proceed inductively on t

for every $t \in \mathcal{T} \setminus \{T\}$ by setting $\Omega := \Omega_1^t, \mathcal{P}_0 := \tilde{\mathcal{P}}_t$ and $\mu := \mathbb{Q}_0 \otimes \ldots \otimes \mathbb{Q}_t$ and constructing $\mathbb{Q}_{t+1}$ as it is done for K in Lemma 4. We can deduce that $\mathbb{Q}_t$ satisfies properties (2a) and (2b). By Fubini's theorem, we conclude that S is a local martingale under the discrete measure $\mathbb{Q} := \mathbb{Q}_0 \otimes \ldots \otimes \mathbb{Q}_{T-1}$. Then (Bouchard and Nutz, 2014, Lemma A.3) guarantees that there exists $\mathbb{Q}'$ equivalent to $\mathbb{Q}$ under which S is a true martingale. As it is shown in Theorem 1, this measure still satisfies the desired properties.

(2)⇒(1): Theorem 7 shows that the market $(\Omega_t \times \Omega_1, \{\mathcal{F}_t, \mathcal{F}_{t+1}\}, S_t, S_{t+1})$ is free of arbitrage $\tilde{\mathcal{P}}_t$-q.s. for every $t \in \mathcal{T} \setminus \{T\}$. Then the result follows with (Bouchard and Nutz, 2014, Theorem 4.5). □

As in Chapter 3, we need the Feller condition to find a measure $\mathbb{Q}$, which has additionally $\mathcal{P}$-full support.

Assumption 4: *We assume that any probability transition kernel $\mathbb{P}_t$ from Ω_t to Ω_1, where $\mathbb{P}_t(\cdot) \in \mathcal{P}_t(\cdot)$, satisfies the weak Feller property for every $t \in \mathcal{T} \setminus \{T\}$.*

Theorem 10: *In a multi-period model with multiple priors, the following are equivalent:*

1. NRA *holds,*
2. *There exists a discrete martingale measure $\mathbb{Q}$ such that for all $t \in \mathcal{T} \setminus \{T\}$,*
   (a) $\mathbb{Q}_t(\omega, \cdot)$ *is a martingale measure $\tilde{\mathcal{P}}_t$-quasi surely,*
   (b) $\mathrm{supp}_{\Delta S_{t+1}(\omega,\cdot)}\mathbb{Q}(\omega, \cdot) = \mathrm{supp}_{\Delta S_{t+1}(\omega,\cdot)}\mathcal{P}_t(\omega)$ *$\tilde{\mathcal{P}}_t$-quasi surely,*
   (c) $\mathrm{supp}_{\Delta S_{t+1}}\mathbb{Q} = \mathrm{supp}_{\Delta S_{t+1}}\mathcal{P}$.

Beweis. We only need to justify that the measure $\mathbb{Q}$ constructed in Theorem 9 has $\mathcal{P}$-full forward support for every $t \in \mathcal{T} \setminus \{T\}$. But this means that $\mathrm{supp}_{\Delta S_{t+1}}\mathbb{Q}_0 \otimes \ldots \otimes \mathbb{Q}_t = \mathrm{supp}_{\Delta S_{t+1}}\tilde{\mathcal{P}}_{t+1}$, which may be shown in the same way as it is done in the proof of Theorem 8. □

3.5 Conclusion

We derived a fundamental theorem of asset pricing under the assumption of multiple-priors or under absence of any prior probability. Our results, and in particular Theorem 9, may serve as a basis to further develop mathematical finance towards quasi-sure analysis on general measurable spaces, where the underlying probability measures are not necessarily dominated by a measure. A promising avenue of future research is the extension of our analysis to a continuous-time setting.

Appendix

Lemma A.1: *Let $\mathcal{M}$ be a family of Borel measures on $\mathbb{R}^d$. Then* $\mathrm{supp}\mathcal{M}$ *is the smallest closed set $A \subseteq \mathbb{R}^d$ such that $\mu(A^c) = 0$ for all $\mu \in \mathcal{M}$. Additionally, $x \in \mathrm{supp}\mathcal{M}$ if and only if for every $\varepsilon > 0$ there exists $\mu \in \mathcal{M}$ with $\mu(B_\varepsilon(x)) > 0$, where $B_\varepsilon(x)$ denotes the ball of radius ε with center x.*

Beweis. First, we note that

$$\begin{aligned}(\mathrm{supp}\mathcal{M})^c &= (\bigcap_{A \subseteq \mathbb{R}^d \text{ closed, } \mu(A^c)=0 \text{ for all } \mu \in \mathcal{M}} A)^c \\ &= \bigcup_{A \subseteq \mathbb{R}^d \text{ closed, } \mu(A^c)=0 \text{ for all } \mu \in \mathcal{M}} A^c \\ &= \bigcup_{A_n \subseteq \mathbb{R}^d \text{ closed, } \mu(A_n^c)=0 \text{ for all } \mu \in \mathcal{M} \text{ and all } n \in \mathbb{N}} A_n^c, \end{aligned} \tag{3.26}$$

where for the second equality we have used de Morgans law and for the last equality we have used that $\mathbb{R}^d$ is strongly Lindelöf. Therefore, we have $\mu((\mathrm{supp}\mathcal{M})^c) \leq \sum_{n=1}^{\infty} \mu(A_n^c) = 0$ for all $\mu \in \mathcal{M}$.

Let $\varepsilon > 0$ and assume $\mu(B_\varepsilon(x)) = 0$ for all $\mu \in \mathcal{M}$ and for some $x \in \mathrm{supp}\mathcal{M}$. Then $\mu(B_\varepsilon(x) \cup \mathrm{supp}\mathcal{M}^c) \leq \mu(B_\varepsilon(x)) + \mu(\mathrm{supp}\mathcal{M}^c) = 0$ for all $\mu \in \mathcal{M}$ and therefore $\mathrm{supp}\mathcal{M} \subseteq B_\varepsilon(x)^c \cap \mathrm{supp}\mathcal{M} \subsetneq \mathrm{supp}\mathcal{M}$, which is a contradiction.
The other direction follows with Lemma A.2. □

Lemma A.2: *Let $\mathcal{P}$ be a family of Borel probability measures on a Polish space X. Then the following holds:*

1. *For all $\mathbb{P} \in \mathcal{P}$ and measurable $C \subseteq X$, $\mathbb{P}(C) > 0$ implies $\mathrm{supp}\mathbb{P} \cap C \neq \emptyset$.*

2. *For all open $\mathcal{O} \subseteq X$, there exists $\mathbb{P} \in \mathcal{P}$ such that $\mathbb{P}(\mathcal{O}) > 0$ if and only if* $\mathrm{supp}\mathcal{P} \cap \mathcal{O} \neq \emptyset$.

Beweis. The first statement is clear with the first statement of Lemma A.1. For the second, we note that,

$$\mathrm{supp}\mathcal{P} = \bigcap_{\substack{A\subseteq X \text{ closed}, \\ \forall \mathbb{P}\in\mathcal{P}:\mathbb{P}(A)=1}} A = \bigcap_{\substack{A\subseteq X \text{ closed}, \\ A\supseteq \cup_{\mathbb{P}\in\mathcal{P}}\mathrm{supp}\mathbb{P}}} A = \overline{\bigcup_{\mathbb{P}\in\mathcal{P}} \mathrm{supp}\mathbb{P}}, \tag{3.27}$$

showing that $\mathrm{supp}\mathcal{P} \cap \mathcal{O} \neq \emptyset$ if and only if there exists $\mathbb{P} \in \mathcal{P}$ satisfying $\mathrm{supp}\mathbb{P} \cap \mathcal{O} \neq \emptyset$. Then the result follows, for example, by Aliprantis and Border (1999). □

The next lemma shows the existence of discrete $\mathcal{P}$-full support measures for any market model.

Lemma A.3 (Existence of discrete $\mathcal{P}$-full forward support measures): *Let $(\Omega, \mathcal{F})$ measurable space, $(X_t)_{t=1,\dots,T}$ be a collection of $\mathcal{F}$-measurable, $\mathbb{R}^d$-valued maps and $\mathcal{P}$ any subset of the set of all probability measures on $(\Omega, \mathcal{F})$. Then there exists a discrete $\mathcal{P}$-full forward support measure $\mathbb{Q}$ with*

$$\mathbb{E}_{\mathbb{Q}}[|X_t|] < \infty \tag{3.28}$$

for all $t \in \{1, \dots, T\}$.

Beweis. Let $F_t := (X_t)^{-1}(\mathrm{supp}_{X_t}\mathcal{P}) \in \mathcal{F}$, and $F := \bigcap_{t=1}^T F_t$ such that $\mathbb{P}(F) = 1$ for all $\mathbb{P} \in \mathcal{P}$. Then we claim,

Claim 10:

$$\mathrm{supp}_{X_t}\mathcal{P} = \overline{\{X_t(\omega) \mid \omega \in F_t\}} = \overline{\{X_t(\omega) \mid \omega \in F\}}$$

Proof of Claim: We start by showing " $\supseteq$ " in both equations. Let $x \in \{X_t(\omega) \mid \omega \in F_t\}$. This means that there exists $\omega \in F_t$ with $x = X_t(\omega)$, which lies in in $\mathrm{supp}_{X_t}\mathcal{P}$. The second follows immediately from $F_t \supseteq F$.

It is now sufficient to show $\mathrm{supp}_{X_t}\mathcal{P} \subseteq \overline{\{X_t(\omega) \mid \omega \in A\}}$ for some A with $\mathbb{P}(A) = 1$ for all $\mathbb{P} \in \mathcal{P}$.

We show this with Claim 7. Let $O \subseteq X$ be open with $O \cap \operatorname{supp}_{X_t}\mathcal{P} \neq \emptyset$. Then by Lemma A.2, there exists $\mathbb{P} \in \mathcal{P}$ such that $\mathbb{P}(X_t \in O) > 0$. As $\mathbb{P}(A) = 1$, it holds $\{X_t \in O\} \cap A \neq \emptyset$, which means that $O \cap \overline{\{X_t(\omega) \mid \omega \in A\}} \neq \emptyset$. □

Hence, as $\operatorname{supp}_{X_t}\mathcal{P}$ is a closed subset of $\mathbb{R}^d$ for all $t \in \{1, \dots, T\}$, we can find $(\omega_{t_k})_{k\in\mathbb{N}} \subseteq F$ such that $\operatorname{supp}_{X_t}\mathcal{P} = \overline{\{X_t(\omega_{t_k}) \mid k \in \mathbb{N}\}}$. Let $(\alpha_{t_k})_{k\in\mathbb{N}} \subseteq (0,1)$ with $\sum_{k=1}^{\infty}\sum_{t=1}^{T}\alpha_{t_k} = 1$ and define $\mathbb{Q}' = \sum_{k=1}^{\infty}\sum_{t=1}^{T}\alpha_{t_k}\delta_{\omega_{t_k}}$. Then it holds $\operatorname{supp}_{X_t}\mathcal{P} = \operatorname{supp}_{X_t}\mathbb{Q}'$ for all $t \in \{1, \dots, T\}$.

Now by (Dellacherie and Meyer, 1982, Thm VII.57), there exists a probability measure $\mathbb{Q}$ equivalent to $\mathbb{Q}'$ such that $\mathbb{E}_{\mathbb{Q}}[|X_t|] < \infty$ for all $t \in \{1, \dots, T\}$. □

Theorem 3. Let $\mathcal{F}_0$ be generated by a partition $\{\mathfrak{a}_n\}_{n=1,\dots,M}$ of Ω, where $M \in \mathbb{N} \cup \{\infty\}$. We can write $\Omega = \bigcup_{n=1}^{M}\mathfrak{a}_n$, where $\mathfrak{a}_n$ is an atom of $\mathcal{F}_0$ for every $n \in \{1, \dots, M\}$.

Claim 11: *The following statements are equivalent:*

i. No $\mathcal{P}$-arbitrage on $(\Omega, \{\mathcal{F}_0, \mathcal{F}\}, S_0, S_1)$.

ii. For all $n \in N$, there is no $\mathcal{P}^n$-arbitrage on $(\Omega^n, \mathcal{F}^n, \Delta S^n)$.

iii. For all $n \in N$, there exists a discrete $\mathcal{P}^n$-full forward support martingale measure $\mathbb{Q}^n$ on $(\Omega^n, \mathcal{F}^n, S_0^n, S_1^n)$.

iv. There exists a discrete $\mathcal{P}$-full forward support martingale measure $\mathbb{Q}$ on $(\Omega, \{\mathcal{F}_0, \mathcal{F}\}, S_0, S_1)$, such that for every $n \in N$, $\mathbb{Q}$ conditional on $\mathfrak{a}_n$ is a $\mathcal{P}^n$-full forward support measure on $(\Omega^n, \mathcal{F}^n, S_0^n, S_1^n)$.

Proof of Claim: (i) ⇒ (ii): We prove the contraposition. Let h_n be a $\mathcal{P}^n$-arbitrage, $n \in N$. Then,

$$\forall \mathbb{P} \in \mathcal{P}^n : \mathbb{P}(h_n \Delta S^n \geq 0) = 1, \tag{3.29}$$

$$\exists \mathbb{P}^* \in \mathcal{P}^n : \mathbb{P}^*(h_n \Delta S^n > 0) > 0. \tag{3.30}$$

Define $h := h_n 1_{\mathfrak{a}_n}$. Then, for every $\mathbb{P} \in \mathcal{P}$ we obtain

$$\mathbb{P}(h\Delta S \geq 0) = \sum_{i=1}^{M}\mathbb{P}(h\Delta S \geq 0 \cap \mathfrak{a}_i) = \sum_{i=1}^{M}\mathbb{P}(\mathfrak{a}_i) = 1, \tag{3.31}$$

where for the second last equality we have used that $h(\omega)\Delta S(\omega) = 0$ for $\omega \in \mathfrak{a}_i$ with $i \neq n$ and $\mathbb{P}(h\Delta S \geq 0 \cap \mathfrak{a}_n) = 0 = \mathbb{P}(\mathfrak{a}_n)$ if $\mathbb{P}(\mathfrak{a}_n) = 0$ and $\mathbb{P}(h\Delta S \geq 0 \cap \mathfrak{a}_n) = \mathbb{P}(\mathfrak{a}_n)\mathbb{P}(h\Delta S \geq 0 \mid \mathfrak{a}_n) = \mathbb{P}(\mathfrak{a}_n)$ if $\mathbb{P}(\mathfrak{a}_n) > 0$ by (3.29). Further, letting $\tilde{\mathbb{P}^*} \in \mathcal{P}$ the corresponding measure for P^*, leads to

$$\tilde{\mathbb{P}^*}(h\Delta S > 0) = \sum_{i=1}^{M} \tilde{\mathbb{P}^*}(h\Delta S > 0 \cap \mathfrak{a}_i) \geq \tilde{\mathbb{P}^*}(\mathfrak{a}_i)\mathbb{P}^*(h\Delta S > 0) > 0, \tag{3.32}$$

where for the last inequality we have used (3.30). Hence, h is a $\mathcal{P}$-arbitrage.

(ii) $\Rightarrow$ (iii): This follows from Theorem 1.

(iii) $\Rightarrow$ (iv): Define a probability measure

$$\mathbb{Q} = \frac{\sum_{n \in N} \mathbb{Q}^n \mathbb{1}_{\mathfrak{a}_n} 2^{-n}}{\sum_{n \in N} 2^{-n}} \tag{3.33}$$

on $(\Omega, \{\mathcal{F}_0, \mathcal{F}\}, S_0, S_1)$, where $\mathbb{Q}^n$ is the corresponding $\mathcal{P}^n$-full forward support martingale measure on $(\Omega^n, \mathcal{F}^n, S_0^n, S_1^n)$ for $n \in N$. Note that $\mathbb{Q}$ still remains a discrete measure by the axiom of countable choice. Clearly, $\mathbb{Q}$ is a martingale measure and for every $n \in N$, $\mathbb{Q}$ restricted to $\mathfrak{a}_n$ has $\mathcal{P}^n$-full forward support. It remains to show that $\mathbb{Q}$ is a $\mathcal{P}$-full forward support measure: For the sake of contradiction, let $x \in \mathrm{supp}_{\Delta S}\mathbb{Q} \setminus \mathrm{supp}_{\Delta S}\mathcal{P}$. Then, there exists $\varepsilon > 0$ such that $(\mathbb{Q} \circ (\Delta S)^{-1})(B_\varepsilon(x)) > 0$ and

$$B_\varepsilon(x) \cap \mathrm{supp}_{\Delta S}\mathcal{P} = \emptyset. \tag{3.34}$$

Since $\mathbb{Q}$ is of the form (3.33), there exists $n \in N$ such that $(\mathbb{Q}^n \circ (\Delta S^n)^{-1})(B_\varepsilon(x)) > 0$ or equivalently $B_\varepsilon(x) \cap \mathrm{supp}_{\Delta S^n}\mathbb{Q}^n \neq \emptyset$. By assumption we know that $\mathrm{supp}_{\Delta S^n}\mathcal{P}^n = \mathrm{supp}_{\Delta S^n}\mathbb{Q}^n$ and therefore we also have $B_\varepsilon(x) \cap \mathrm{supp}_{\Delta S^n}\mathcal{P}^n \neq \emptyset$. But the latter implies that there exists $\mathbb{P}^n \in \mathcal{P}^n$ with corresponding $\mathbb{P} \in \mathcal{P}$ satisfying

$$0 < \mathbb{P}^n((\Delta S^n)^{-1}(B_\varepsilon(x))) = \mathbb{P}((\Delta S)^{-1}(B_\varepsilon(x)) \mid \mathfrak{a}_n), \tag{3.35}$$

i.e. $0 < \mathbb{P}((\Delta S)^{-1}(B_\varepsilon(x) \cap \mathfrak{a}_n) \leq \mathbb{P}((\Delta S)^{-1}(B_\varepsilon(x)))$, which is a contradiction to (3.34). Thus, we have shown $\mathrm{supp}_{\Delta S}\mathbb{Q} \subseteq \mathrm{supp}_{\Delta S}\mathcal{P}$. In a similar way we can show that $\mathrm{supp}_{\Delta S}\mathcal{P} \subseteq \mathrm{supp}_{\Delta S}\mathbb{Q}$ must hold.

(iv) $\Rightarrow$ (i): Assume (iv). To achieve a contradiction, assume that there exists a $\mathcal{P}$-arbitrage h.

Then,

$$\forall \mathbb{P} \in \mathcal{P} : \mathbb{P}(h \cdot \Delta S \geq 0) = 1, \tag{3.36}$$

$$\exists \mathbb{P}' \in \mathcal{P} : \mathbb{P}'(h \cdot \Delta S > 0) > 0. \tag{3.37}$$

We obtain

$$0 < \mathbb{P}'(h \cdot \Delta S > 0) = \sum_{n=1}^{M} \mathbb{P}'(h \cdot \Delta S > 0 \cap \mathfrak{a}_n). \tag{3.38}$$

Therefore, there exists $n \in \mathbb{N}$ such that $\mathbb{P}'(h \cdot \Delta S > 0 \cap \mathfrak{a}_n) > 0$. We have $h(\omega)\Delta S(\omega) = h_n \Delta S(\omega)$ for all $\omega \in \mathfrak{a}_n$ and some $h_n \in \mathbb{R}^d$. Hence $\mathbb{P}'(\cdot \mid \mathfrak{a}_n)(\Delta S \in H_{>}^{h_n}) > 0$ and

$$\mathrm{supp}_{\Delta S} \mathbb{P}'(\cdot \mid \mathfrak{a}_n) \cap H_{>}^{h_n} \neq \emptyset. \tag{3.39}$$

By (3.36) we have for all $\mathbb{P}^n \in \mathcal{P}^n$ with corresponding $\mathbb{P} \in \mathcal{P}$,

$$\mathbb{P}^n(\Delta S^n \in H_{\geq}^{h_n}) = \mathbb{P}(h_n \cdot \Delta S^n \geq 0 \mid \mathfrak{a}_n) = \mathbb{P}(h \cdot \Delta S \geq 0 \mid \mathfrak{a}_n) = 1$$

and hence, using (3.27),

$$\mathrm{supp}_{\Delta S^n} \mathcal{P}^n = \overline{\bigcup_{\mathbb{P}^n \in \mathcal{P}^n} \mathrm{supp}_{\Delta S^n} \mathbb{P}^n} \subseteq H_{\geq}^{h_n}. \tag{3.40}$$

By assumption, there exists a discrete martingale measure $\mathbb{Q}$ for the market model $(\Omega, \{\mathcal{F}_0, \mathcal{F}\}, \Delta S)$, such that $\mathrm{supp}_{\Delta S^{n'}} \mathbb{Q}(\cdot \mid \mathfrak{a}_{n'}) = \mathrm{supp}_{\Delta S^{n'}} \mathcal{P}^{n'}$ for every $n' \in N$. Hence by (3.40),

$$\mathrm{supp}_{\Delta S^n} \mathbb{Q}(\cdot \mid \mathfrak{a}_n) \subseteq H_{\geq}^{h_n}, \tag{3.41}$$

and by (3.39)

$$\mathrm{supp}_{\Delta S^n} \mathbb{Q}(\cdot \mid \mathfrak{a}_n) \cap H_{>}^{h_n} \neq \emptyset. \tag{3.42}$$

Consequently, for $h' := h \mathbb{1}_{\mathfrak{a}_n}$ we have $0 = \mathbb{E}_{\mathbb{Q}}[\mathbb{E}_{\mathbb{Q}}[\Delta S \mid \mathcal{F}_0] \cdot h'] = \mathbb{E}_{\mathbb{Q}}[\mathbb{E}_{\mathbb{Q}}[h' \cdot \Delta S \mid \mathcal{F}_0]] = \mathbb{E}_{\mathbb{Q}}[h' \cdot \Delta S] > 0$, which is a contradiction and finishes the proof of Claim 11. $\square$

Claim 11 implies the statements of the theorem. □

THEOREM A.1: *Let Ω be a Polish space and ΔS be continuous. Then the following are equivalent in the one period model for constant S_0:*

1. NRA *holds.*
2. *There exists a discrete $\mathcal{P}$-full support martingale measure $\mathbb{Q}$.*

Beweis. The proof follows the same steps as the proof of Theorem 1, by noticing that $\{\omega \in \Omega \mid h \cdot \Delta S(\omega) \geq 0\}$ is closed and $\{\omega \in \Omega \mid h \cdot \Delta S(\omega) > 0\}$ is open by the continuity of ΔS.

□

LEMMA A.4: *In the one period framework with possibly measurable initial price, the following holds:*

(a) Let ΔS be continuous. Then a $\mathcal{P}$-full support measure is also a $\mathcal{P}$-full forward support measure.

(b) Let ΔS be open. Then a $\mathcal{P}$-full forward support measure is also a $\mathcal{P}$-full support measure.

In particular, if ΔS is continuous and open, then $\mathbb{Q}$ being a $\mathcal{P}$-full support measure is equivalent with $\mathbb{Q}$ being a $\mathcal{P}$-full forward support measure.

Beweis. Assume in (a) that there exists $x \in \mathrm{supp}_{\Delta S}\mathbb{Q} \setminus \mathrm{supp}_{\Delta S}\mathcal{P}$ for some $\mathcal{P}$- full support measure $\mathbb{Q}$ and continuous ΔS. Then choose $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq (\mathrm{supp}_{\Delta S}\mathcal{P})^c$. Then $B_\varepsilon(x) \cap \mathrm{supp}_{\Delta S}\mathbb{Q} \neq \emptyset$ implies by Lemma A.2 $\mathbb{Q}((\Delta S)^{-1}(B_\varepsilon(x)) > 0$ and hence $\mathrm{supp}\mathbb{Q} \cap (\Delta S)^{-1}(B_\varepsilon(x)) \neq \emptyset$. However, we have $\mathbb{P}((\Delta S)^{-1}(B_\varepsilon(x)) = 0$ for every $\mathbb{P} \in \mathcal{P}$ and hence by continuity of ΔS and Lemma A.2, we have $\mathrm{supp}\mathcal{P} \cap (\Delta S)^{-1}(B_\varepsilon(x)) = \emptyset$, which gives the contradiction. The other direction is done in the same way.
For (b) we assume that there exists $\omega \in \mathrm{supp}\mathbb{Q} \setminus \mathrm{supp}\mathcal{P}$ for some $\mathcal{P}$-full forward support measure $\mathbb{Q}$ and open ΔS. Then choose $\varepsilon > 0$ such that $B_\varepsilon(\omega) \subseteq (\mathrm{supp}\mathcal{P})^c$. Then $B_\varepsilon(\omega) \cap \mathrm{supp}\mathbb{Q} \neq \emptyset$ implies $\mathbb{Q}((\Delta S)^{-1}((\Delta S)(B_\varepsilon(\omega)))) = \mathbb{Q}(B_\varepsilon(\omega)) > 0$ and $\mathrm{supp}_{\Delta S}\mathbb{Q} \cap \Delta S(B_\varepsilon(\omega)) \neq \emptyset$. However, for all $\mathbb{P} \in \mathcal{P}$ we find $\mathbb{P}((\Delta S)^{-1}((\Delta S)(B_\varepsilon(\omega)))) = \mathbb{P}(B_\varepsilon(\omega)) = 0$ and hence by the openness of ΔS we have $\mathrm{supp}_{\Delta S}\mathcal{P} \cap \Delta S(B_\varepsilon(\omega)) = \emptyset$, which is the contradiction. The other direction follows similar. □

LEMMA A.5: *Let X be a continuous random variable on a Polish space Ω_1, $\mathcal{P}$ be a set of probability measures on Ω_1 and $(\omega_k)_{k\in\mathbb{N}} \subseteq \Omega_1$. Then,*

$$\mathrm{supp}\mathcal{P} = \overline{\{\omega_k\}_{k\in\mathbb{N}}} \;\Rightarrow\; \mathrm{supp}_X\mathcal{P} = \overline{\{X(\omega_k)\}_{k\in\mathbb{N}}}.$$

Beweis. Setting $\mathbb{Q} = \sum_{k=1}^{\infty} a_k\delta_{\omega_k}$ for some $a_k > 0$ for all k, we find $\mathrm{supp}\mathbb{Q} = \overline{\{\omega_k\}_{k\in\mathbb{N}}} = \mathrm{supp}\mathcal{P}$, and using Lemma A.4, we conclude $\overline{\{X(\omega_k)\}_{k\in\mathbb{N}}} = \mathrm{supp}_X\mathbb{Q} = \mathrm{supp}_X\mathcal{P}$. □

PROPOSITION A.1: *There exists a $\mathcal{P}$-arbitrage in the multi-period market model if and only if there exists $t \in \{1,\dots,T\}$ and a $\mathcal{F}_{t-1}$-measurable map $h : \Omega \to \mathbb{R}^d$ such that*

$$\forall\mathbb{P} \in \mathcal{P}:\; \mathbb{P}(h\cdot\Delta S_t \geq 0) = 1, \tag{3.43}$$

$$\exists\mathbb{P} \in \mathcal{P}: \mathbb{P}(h\cdot\Delta S_t > 0) > 0. \tag{3.44}$$

Beweis. The proof is similar to Föllmer and Schied (2011), Prop. 5.11. Let ξ be an $\mathcal{P}$-arbitrage and V its gain process with $V_0 = 0$. Define

$$t := \min\{k \mid \mathbb{P}(V_k \geq 0) = 1 \;\text{ for all } \mathbb{P}\in\mathcal{P},\; \text{and } \mathbb{P}(V_k > 0) > 0 \text{ for some } \mathbb{P}\in\mathcal{P}\}. \tag{3.45}$$

Then, by assumption $t \leq T$ and we have the following two cases

1. either $\forall\mathbb{P}\in\mathcal{P} : \mathbb{P}(V_{t-1} \geq 0) = 1$ and $\mathbb{P}(V_{t-1} > 0) = 0$
2. or $\exists\; \mathbb{P}\in\mathcal{P},\; \mathbb{P}(V_{t-1} \geq 0) < 1$,

since t was chosen minimal. Hence, in the first case, it follows that $\mathbb{P}(V_{t-1} = 0) = 1$ for all $\mathbb{P}\in\mathcal{P}$ and therefore

$$\forall\mathbb{P}\in\mathcal{P} : \mathbb{P}(\xi_t\cdot(S_t - S_{t-1}) = V_t - V_{t-1} = V_t) = \mathbb{P}(V_{t-1} = 0) = 1. \tag{3.46}$$

Thus, $h := \xi_t$ satisfies (3.43) and (3.44). In the second case, we take $h := \xi_t 1_{\{V_{t-1}<0\}}$, which is $\mathcal{F}_{t-1}$-measurable and satisfies

$$\begin{aligned} h(\omega)(S_t(\omega) - S_{t-1}(\omega)) &= (V_t(\omega) - V_{t-1}(\omega))1_{\{V_{t-1}<0\}}(\omega) \\ &\geq -V_{t-1}(\omega)1_{\{V_{t-1}<0\}}(\omega) \geq 0, \end{aligned} \tag{3.47}$$

for every $\omega \in \Omega$. By case 2, there exists $\mathbb{P} \in \mathcal{P}$ such that $\mathbb{P}(V_{t-1} < 0) > 0$, hence the random variable on the right hand side of (3.47) is strictly positive with positive probability for some $\mathbb{P} \in \mathcal{P}$. This proves necessity.

Next, we prove sufficiency. For t and h as in the statement of the proposition and satisfying (3.43) and (3.44), define

$$\xi_s = \begin{cases} h & \text{, if } s = t \\ 0 & \text{, else.} \end{cases}$$

Then, ξ is a trading strategy which is also an $\mathcal{P}$-arbitrage. $\square$

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Kapitel 4

Simulating Negative Feed-Back Effects in Financial Systems

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This paper was presented at the UZH Seminar in Contract Theory and Banking.

Abstract: In a financial system of banks, which are linked through interbank lending, we postulate that indirect spillover effects, such as funding liquidity dry-ups and asset fire sales, can play a crucial role. Considering only direct effects, i.e. bank defaults, underestimates the extent of a crisis, thereby showing the importance of indirect spillover effects. To model funding liquidity, we allow banks to access the interbank lending market and as a second channel, to obtain liquidity from a repo transaction. Additionally, the developed framework enables us to impose exogenous shocks to the financial system, for instance bank defaults, asset price deteriorations, higher margins and haircuts, and a sharp increase of the short-term interest rate. Consequently, this paper builds a unified framework that can enrich the toolbox used by national banks and the European central bank.

4.1 Introduction

What is systemic risk in finance? There is no consensus on the definition, but what we are talking about is the risk of a collapse of a large part of the financial system and the impact on the real economy. Following the 2007 – 2008 subprime mortgage crisis, the discussions on systemic risk and the *too big to fail* problematic have flourished. These topics are of great interest to financial regulators and have yet to be fully understood. This uncertainty entails the necessity of stress tests, which simulate the evolvement of shocks through a financial system and gauge the stability of the given entity.

We propose a stress test environment which encompasses an appropriate time frame of a few weeks, e.g. two weeks, and consists of ten banks at its core which are linked through overnight interbank lending activities. The terminology *bank* is employed to refer to any financial institution like a commercial bank, a private bank, a retail bank, or another financial intermediary. The model design can be easily extended to include more than ten banks, but considering a few number of banks copes with our purpose. Also, with regard to the fact that in the United States (US) 60% of all US commercial bank assets are held only by the largest six banks, a model which includes the ten largest banks already covers a large part of the banking sector.[1] We assume each bank's overnight interbank liabilities, and hence also the corresponding asset values, to be constant, which is justified by the fact that in times of (financial) distress banks are likely to have to roll over their overnight interbank liabilities over the short time frame we consider. In our model, banks hold fully liquid assets, e.g. cash, and additionally are allowed to obtain liquidity from entering a repurchase agreement (repo) transaction by using their repo assets as collateral. The liabilities of a bank are decomposed into overnight interbank–, short-term (e.g. three month)–, repo–, deposit–, and long-term (e.g. ten year) liabilities. The latter remains unaffected in our model due to its long maturity and the short time frame we consider. Similarly, the assets of a bank are decomposed into different classes, such as fully liquid–, overnight interbank–, repo–, and reverse repo assets, and further (risky) assets, e.g. deposits, mortgage loans, corporate loans, derivatives, or fixed assets. Repo assets, generally a subset of the collateral assets, are those assets which a bank can use as collateral to obtain liquidity by entering a repo transaction. For

[1] The same qualitative reasoning holds true for the European banking sector where for instance the HSBC Holdings plc exhibits more than twice as much assets as the UBS AG in 2013.

simplicity's sake we assume in our framework that the collateral and repo assets, respectively, coincide. The conducted classification yields overall a rich and flexible structure of nine (five) different asset (liability) classes.

Through the interbank lending linkages, the default of one bank can cause the default of further banks, and this effect is referred to as *the direct effect.* To keep track of the evolvement of a shock through the system, we model the balance sheet of each bank on a daily basis. If the top tier capital ratio (TTCR) of a bank, which is the ratio of equities minus assets over risk-weighted assets,[2] falls below a certain threshold value (e.g. 7%), we say that this bank defaults. Along with the direct effect, the so-called *indirect effects* can take place. These effects are, among others, increasing margins and haircuts in the repo market, increasing interest rates on short-term (unsecured) lending, and fire sales. At $t = 1$ we model indirect effects independently of direct effects. However, these indirect effects are crucial drivers for recurring direct effects for $t > 1$. Our model allows to shock the system in various ways, resulting in manifold triggers of a crisis. All shocks occur at time $t = 1$ and can be, among others, an arbitrary large asset price deterioration, an immediate default of a bank, or an arbitrary increase of aggregate and/or bank specific margins and haircuts. Moreover, by imposing two different liquidity measures which are distinct with respect to the time horizon, we can assess a bank's overnight and short-term liquidity, respectively.

Our paper is based on the findings of Brunnermeier (2009) and Afonso et al. (2011). Brunnermeier (2009) argues that fire sales were an amplifying effect to the 2007 – 2008 financial crisis. In Afonso et al. (2011), empirical evidence is given for a funding dry-up in the interbank and repo market, respectively, after the bankruptcy of Lehman Brothers in 2008. Contrary to the prevailing opinion, the funding market did not freeze completely, but was merely in distress. Some models try to explain these funding dry-ups through liquidity hoarding, whereby the terminology *liquidity hoarding* is used to refer to the action that banks significantly reduce their lending activities on the interbank market during times of distress, irrespectively of their counterparty quality. As it is shown in Afonso et al. (2011), there is no empirical evidence for liquidity hoarding but rather counterparty concerns played a more important role in explaining the reduced liquidity and led to high costs of borrowing for weak banks.

[2]We refer to equation (4.1) for the exact mathematical representation.

The paper is structured as follows. 4.2 presents our model in detail. In 4.3 we list our numerical results. Finally, 4.4 concludes. The tables and figures are given in the appendices.

4.2 The Model Design

In this section we introduce our dynamic model for the financial system and elaborate the corresponding effects.

4.2.1 The Balance Sheet Structure

Throughout this paper let $\mathbb{R}_{\geq 0}$ be the non-negative real numbers, i.e. $\mathbb{R}_{\geq 0} := [0, \infty)$. Our model contains $N \in \mathbb{N}$ financial institutions, each of them indexed by $n \in [N]$, where for any natural number $X \in \mathbb{N}$ we define $[X] := \{1, \ldots, X\}$. We denote by $T \in \mathbb{N}$ our finite time horizon with current time $t \in [T] \cup \{0\}$. For simplicity's sake, we call each financial institution a bank, although it can be a commercial bank, a private bank, a retail bank, or another financial intermediary.[3] Notice, we do not incorporate *unregulated* shadow banks such as hedge funds, money market mutual funds, and investment banks which are not subject to the international Basel III requirements. This distinction is of high importance since we work with a capital ratio introduced by Basel III. To specify the liability structure in the interbank market, we assume a matrix $(L^{\mathrm{IB}}_{nm,0})_{n,m\in[N]} \in \mathbb{R}^{N\times N}_{\geq 0}$ with zeros on the diagonal and non-negative off-diagonal entries, where $L^{\mathrm{IB}}_{nm,0}$ represents the book value at time $t = 0$ of the overnight interbank liability that bank n has to pay to bank m at time $t = 1$. Moreover, we let

$$L^{\mathrm{IB}}_{n,0} = \sum_{m=1}^{N} L^{\mathrm{IB}}_{nm,0}$$

be the total interbank liabilities of bank $n \in [N]$ at time $t = 0$, and similarly we let

$$A^{\mathrm{IB}}_{n,0} = \sum_{m=1}^{N} L^{\mathrm{IB}}_{mn,0}$$

be the book value at time $t = 0$ of total interbank assets that bank n is going to receive at time $t = 1$. Further, for every bank $n \in [N]$ we assume the bank n's initial fully liquid asset value to

[3] This notational use is in line with the related literature (see, e.g., Gai and Kapadia (2010) and Gai et al. (2011)).

be $A_{n,0}^{\mathrm{LIQ}}$. Moreover, a bank holds assets $A_{n,0}^{\mathrm{C}}$ which can be used as collateral to obtain liquidity from the repo market, and is also equipped with reverse repo assets $A_{n,0}^{\mathrm{RR}}$ which is a common form of collateralized lending whereas a repo itself is simply a collateralized loan.[4] We emphasize that the repo market plays an essential role in the interbank market. From Figure 4.1 we can observe an increasing outstanding amount of repo liabilities over the last fourteen years even though the peak was achieved during the 2008 – 2009 financial crisis. This indeed implies that a well-functioning repo market is pivotal for a viable interbank market. Eventually, banks are endowed with deposits $A_{n,0}^{\mathrm{D}}$, residential mortgages $A_{n,0}^{\mathrm{M}}$, corporate loans $A_{n,0}^{\mathrm{CL}}$, derivatives $A_{n,0}^{\mathrm{DV}}$, and a given amount of fixed assets $A_{n,0}^{\mathrm{F}}$.

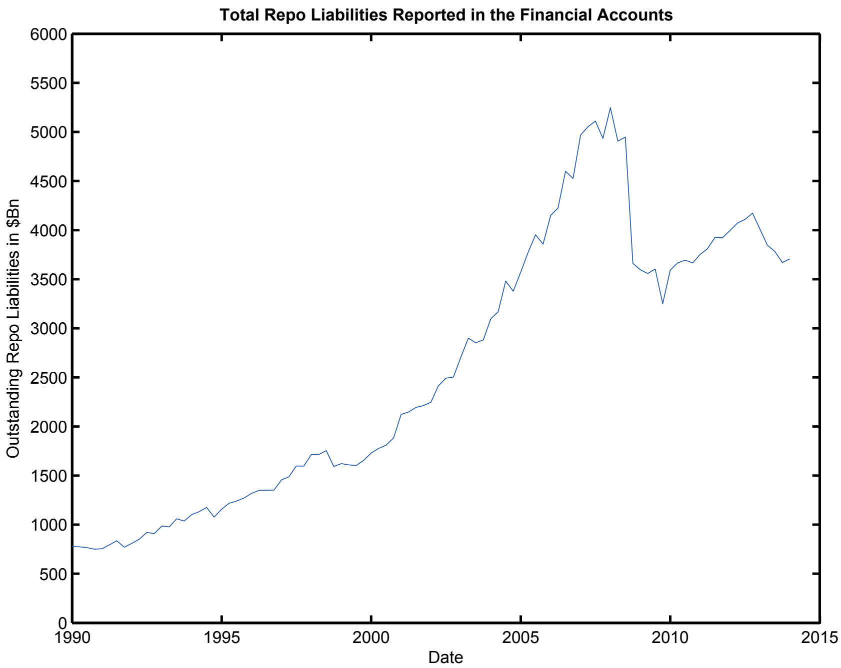

Abbildung 4.1: We plot the total repo liabilities in \$bn reported in the financial accounts of the United States over time. *Data Source*: Financial Accounts of the United States.

Remark 5: Usually deposit assets are subject to deposit protection rules. Introducing this additional degree of freedom is, among others, country– and institution-specific and we leave this task to the entity which uses our framework. During the crisis, banks which offered an insurance on deposits were able to increase the inflow of deposits by increasing the interest rate on it. This

[4]Common types of collateral include US treasury securities, agency securities, mortgage-backed securities, corporate bonds, equity, and customer collateral. Moreover, typical cash providers consist of money market mutual funds, insurance companies, corporations, municipalities, central banks, securities lenders, and commercial banks whereas the security providers are decomposed of security lenders, hedge funds, levered accounts, central banks, commercial banks, and insurance companies.

is an effect which should be taken care of, but we leave this task as an interesting avenue of future research.

Incorporating derivatives into the balance sheet structure is crucial since this position can absorb up to one third of a bank's asset structure.[5] Here we use the convention of financial accounting standards which require that an entity recognizes all derivatives as either assets or liabilities in the statement of financial position. Consequently, the total assets at time $t = 0$ are given by

$$A_{n,0} = A^{\text{LIQ}}_{n,0} + A^{\text{IB}}_{n,0} + \underbrace{p^{\text{C}}_0 V^{\text{C}}_{n,0}}_{\equiv A^{\text{C}}_{n,0}} + \underbrace{p^{\text{RR}}_0 V^{\text{RR}}_{n,0}}_{\equiv A^{\text{RR}}_{n,0}} + A^{\text{D}}_{n,0} + \underbrace{p^{\text{M}}_0 V^{\text{M}}_{n,0}}_{\equiv A^{\text{M}}_{n,0}} + \underbrace{p^{\text{CL}}_0 V^{\text{CL}}_{n,0}}_{\equiv A^{\text{CL}}_{n,0}} + \underbrace{p^{\text{DV}}_0 V^{\text{DV}}_{n,0}}_{\equiv A^{\text{DV}}_{n,0}} + \underbrace{p^{\text{F}}_0 V^{\text{F}}_{n,0}}_{\equiv A^{\text{F}}_{n,0}},$$

where p^{C}_0 $(V^{\text{C}}_{n,0})$, p^{RR}_0 $(V^{\text{RR}}_{n,0})$, p^{M}_0 $(V^{\text{M}}_{n,0})$, p^{CL}_0 $(V^{\text{CL}}_{n,0})$, p^{DV}_0 $(V^{\text{DV}}_{n,0})$, and p^{F}_0 $(V^{\text{F}}_{n,0})$ are the initial price (volume) of the collateral asset, the initial price (volume) of the reverse repo asset, the initial price (volume) of the residential mortgage loan, the initial price (volume) of the corporate loan, the initial price (volume) of the derivative, and the initial price (volume) of the fixed asset[6], respectively. This structure is an extension of Gai et al. (2011) who propose a similar setting but do not model explicitly the deposits, the residential mortgages, the corporate loans, and the derivatives.

Remark 6: Blatantly this setting is a simplified version of the reality since we assume *one* price for each asset class. Nonetheless, this framework enables an easy way to shock the different asset classes by shocking the corresponding asset prices.

We assume that banks pass the funds $A^{\text{D}}_{n,t}$, for all $t \in [T]$ and all $n \in [N]$, on to borrowers and receive interest on the loans. Hence, it holds that

$$A^{\text{D}}_{n,t} = (1 + R^{\text{A}}_t) A^{\text{D}}_{n,t-1},$$

where $R^{\text{A}}_t \in [0, 1]$ denotes the interest rate of the loans. This business model is a common way for banks to derive profits from customer deposits.

[5] We would like to thank Eric Jondeau for pointing this out.

[6] For the sake of simplicity, we consider the fixed assets of a bank to be only one asset class. However, the subdivision of fixed assets into more asset classes is straightforward.

We denote by $A_{n,t}$ the book value of total assets of bank n and by $L_{n,t}$ its total liabilities at time $t \in [T] \cup \{0\}$. Hence, the equity book value at time t of bank n is

$$E_{n,t} = A_{n,t} - L_{n,t}.$$

Finally, we introduce the following top tier capital ration (TTCR) defined as

$$\mathrm{TTCR}_{n,t} := \frac{E_{n,t}}{\mathrm{RWA}_{n,t}}, \tag{4.1}$$

where we denote by

$$\mathrm{RWA}_{n,t} := w_1 A^{\mathrm{LIQ}}_{n,t} + w_2 A^{\mathrm{IB}}_{n,t} + w_3 A^{\mathrm{C}}_{n,t} + w_4 A^{\mathrm{RR}}_{n,t} + w_5 A^{\mathrm{D}}_{n,t} + w_6 A^{\mathrm{M}}_{n,t} + w_7 A^{\mathrm{CL}}_{n,t} + w_8 A^{\mathrm{DV}}_{n,t} + w_9 A^{\mathrm{F}}_{n,t} \tag{4.2}$$

the risk-weighted assets with $(w_i)_{i=1,\dots,9} \in [0,1]^9$.

Definition 12 (**Solvency**): *Fix* $\phi \in [0,1]$. *We say that a bank* $n \in [N]$ *is* solvent (insolvent) *at time* $t \in [T]$, *if it satisfies (does not satisfy) equation*

$$\mathrm{TTCR}_{n,t} > \phi. \tag{4.3}$$

Additionally, we assume that each bank tries to keep its TTCR above a target threshold $\phi_{\text{target}} \in [0,1]$, i.e. each bank $n \in [N]$ wants to fulfill the inequality

$$\mathrm{TTCR}_{n,t} > \phi_{\text{target}} \tag{4.4}$$

at all times $t \in [T]$,[7] where $\phi_{\text{target}} \in [0,1]$ with $\phi < \phi_{\text{target}}$. A typical value in the Basel III framework is $\phi_{\text{target}} \in [0.07, 0.095]$.[8]

Remark 7: The top tier capital ratio introduced in (4.1) primarily serves us to distinguish between solvent and insolvent banks and is not to be confused with the TTCR used by regulators, since the latter is calculated with respect to more factors.

[7] Our notation implies that at time $t = 0$ no defaults occur.

[8] The European Banking Authority (EBA) used $\phi = 0.055$ in 2014 for the EU-wide stress test of 123 banks.

Remark 8: Very recently the Financial Stability Board (FSB) suggested a proposal for a common international standard on Total Loss-Absorbing Capacity (TLAC) for global systemic banks which is a complement to the Basel III capital requirements. The committee suggests 16 – 20 percent of risk-weighted assets and at least twice the Basel III tier 1 leverage ratio ($\neq$TTCR) requirement. We do not incorporate the TLAC in our analysis since the proposal has to be refined (expected final version in 2015).

4.2.2 Direct Spill-Over Effect

When a bank defaults on its debts, other banks may suffer from a loss in assets if they have a credit exposure on the defaulting one. However, the loss is generally not the total amount of loan to the defaulting bank, but only a certain percentage. With practitioners' vocabulary, the percentage of loss is $1 - r$, with r being the recovery rate. The recovery rate varies a lot in different default cases: 95% as estimated by Kaufman (1994) for the Continental Illinois case and 28% in the case of Lehman Brothers' collapse in September, 2008 (cf. Fleming and Sarkar (2014)). In the short time frame under consideration, a bank is highly unlikely to obtain the recovered amount from a defaulting bank. For this reason, one should set the recovery rate equal to zero. However, in terms of book values, a positive recovery rate is reasonable, which is why we consider both cases. We assume upon default, the recovered amount is immediately paid to the creditor in the form of risk-free asset. In reality, these interbank loans are part of the assets of the defaulting bank $n \in [N]$ when it files for bankruptcy and the assets will be sold to other investors to pay back the debt holders as much as possible. As a consequence, the creditors of the defaulting bank still have to pay for their loans from the defaulting banks if there is any. Explicitly, if bank $n \in [N]$ defaults at time $t \in [T]$, then the interbank loans $(L^{\mathrm{IB}}_{mn,t})_{m\in[N]}$ will be settled, but not in full. For a bank $m \in [N]$, the creditor of bank n, it will receive an immediate payment rL^{IB}_{mn} in the form of a risk-free asset. The loss would reduce bank m's TTCR, and potentially cause bank m's default too. We refer to this effect as a *direct spillover effects.*

Explicitly, we have the following setup: For $t \in [T] \cup \{0\}$, we denote by $D_t \subseteq [N]$ the set of all banks that default at time t with the convention $D_0 = \emptyset$, and we let D^c_t be its complement.

For $m \in D_t$ we set

$$L^{\mathrm{IB}}_{mn,t} = 0 \quad \forall n \in [N],$$

and for $n \in D^c_t$ we set

$$A^{\mathrm{LIQ}}_{n,t} = A^{\mathrm{LIQ}}_{n,t-1} + r \sum_{m \in D_t} L^{\mathrm{IB}}_{mn,t-1}, \tag{4.5}$$

where the second term on the right hand side of (4.5) is the recovered amount of bank n from all defaulting banks at time $t \in [T]$. On the contrary, the interbank loans $(L^{\mathrm{IB}}_{mn,t})_{m\in[N]}$ to the defaulting bank n remain unaffected and still appear on the balance sheets of the non-defaulting banks at time $t \in [T]$.

Remark 9: It is quite natural to think about netting the interbank loan matrix: If at time $t \in [T] \cup \{0\}$ a bank $n \in [N]$ owes bank $m \in [N]$ one million USD and the bank m owes bank n one million USD as well, then the two lending positions compensate with each other perfectly. This results in a procedure of netting the interbank loan matrix by comparing the elements $L^{\mathrm{IB}}_{nm,t}$ and $L^{\mathrm{IB}}_{mn,t}$, replacing the smaller value by zero and the larger one by the difference. However, it is important to point out that the interbank loan matrix cannot be 'netted' artificially in the previous manner, since the two scenarios described for the simple example are far from being equivalent when a default happens. If there are no lending activities between the banks, the bankruptcy of bank m does not affect the bank n, but in the original scenario, the bank n would suffer from the bank m's default, and bank n's loss incurred by the default depends both on the seniority of interbank loan and the recovery rate of the default.

4.2.3 Funding Liquidity Dry-Up

Besides the interbank liabilities, a bank also has liabilities outside the interbank system and we divide this liabilities into short-term liabilities $L^{\mathrm{S}}_{n,t}$ and long-term liabilities $L^{\mathrm{L}}_{n,t}$. We do not consider overnight interbank liabilities as a part of short-term debt, but rather think of short-term debts as being a liability with a three month maturity. The total value of liabilities at time

$t \in [T] \cup \{0\}$ of any bank $n \in D_t^c$ is therefore

$$L_{n,t} = L_{n,t}^{\mathrm{IB}} + L_{n,t}^{\mathrm{R}} + L_{n,t}^{\mathrm{S}} + L_{n,t}^{\mathrm{D}} + L_{n,t}^{\mathrm{L}},$$

where $L_{n,t}^{\mathrm{R}}$ is the repo liability and $L_{n,t}^{\mathrm{D}}$ denotes the deposit liability of bank n at time t. We impose the reasonable assumption that the interbank liabilities $L_{n,t}^{\mathrm{IB}}$ are rolled over, i.e.

$$L_{n,t}^{\mathrm{IB}} = L_{n,0}^{\mathrm{IB}}$$

for all $t \in [T]$ and all $n \in [N]$, and neglect overnight interest rate fluctuations. We assume that the short-term debt of any bank has to be rolled over daily

$$L_{n,t}^{\mathrm{S}} = (1 + R_t^{\mathrm{S}}) L_{n,t-1}^{\mathrm{S}},$$

thereby facing the risk that the short-term interest rate $R_t^{\mathrm{S}} \in [0,1]$ might increase, and additionally we assume that the long-term debt is constant over time, i.e.

$$L_{n,t}^{\mathrm{L}} = L_{n,0}^{\mathrm{L}}$$

for all $t \in [T]$ and all $n \in [N]$. The interest rate on unsecured short-term debt can be fixed. If in a crisis a bank is close to maturity of this debt, it is likely to have to roll over this debt at the new increased interest rate. Since the time-to-maturity of short-term debt is different for each bank, the increased interest rate might affect some banks but not others. We generally account for this effect by assuming a floating interest rate. Despite the assumption of floating interest rates, it reflects the liquidity of a bank. Eventually, since banks take deposits from savers and pay interest on these accounts, the deposit liabilities are evolving accordingly:

$$L_{n,t}^{\mathrm{D}} = (1 + R_t^{\mathrm{D}}) L_{n,t-1}^{\mathrm{D}},$$

for all $t \in [T]$ and all $n \in [N]$ given the deposit rate $R_t^{\mathrm{D}} \in [0,1]$. We also impose the incentive mechanism $R_t^{\mathrm{D}} < R_t^{\mathrm{A}}$ for all $t \in [T]$ which assures that the banks make profits since they are derived from the spread between the rate they pay for funds and the rate they receive from borrowers.

Using the framework of 4.2.1 and the results of Gai et al. (2011), the repo liability of bank $n \in [N]$ at time $t \in [T]$ is given by

$$L^{\mathrm{R}}_{n,t} = (1 + R^{\mathrm{R}}_{n,t}) \left((1 - H_t - H_{n,t})\, p^{\mathrm{C}}_t V^{\mathrm{C}}_{n,t} + \frac{(1 - H_t - H_{n,t})}{(1 - H_t)} p^{\mathrm{RR}}_t V^{\mathrm{RR}}_{n,t} \right),$$

where $R^{\mathrm{R}}_{n,t} \in [0,1]$ denotes the bank-specific repo rate, $H_t \in [0,1)$ is the aggregate haircut, and $H_{n,t} \in [0,1]$ stands for the bank-specific haircut at time t such that $H_t + H_{n,t} \leq 1$ holds. The first expression $(1 - H_t - H_{n,t})\, p^{\mathrm{C}}_t V^{\mathrm{C}}_{n,t}$ comes from the pledging of collateral assets whereas the second expression $\frac{(1-H_t-H_{n,t})}{(1-H_t)} p^{\mathrm{RR}}_t V^{\mathrm{RR}}_{n,t}$ is the amount raised from rehypothecating collateral (see also Lo (2011)). The amount $\frac{p^{\mathrm{RR}}_t V^{\mathrm{RR}}_{n,t}}{(1-H_t)}$ is due to the fact that reverse repo transactions are secured with collateral that commands the same aggregate haircuts on $A^{\mathrm{C}}_{n,t}$ (see Gai et al. (2011)). To illustrate intuitively how these transactions work, we provide a short example.

Example 4: *Assume bank A has an amount X of collateral value at time $t \in [T]$. We call the counterparty of bank A 'the market'. If bank A enters a repo transaction with the market using the collateral X, it receives the amount $(1 - H_t - H_{\mathrm{A},t})\, X$ of cash. At the same time bank A enters a reverse repo transaction with the market, where the market wants to get Y amount of cash. Therefore, the market has to provide $\frac{Y}{(1-H_t)}$ of collateral to bank A. Now, bank A can reuse the collateral with value $\frac{Y}{(1-H_t)}$ to obtain additional cash of $\frac{(1-H_t-H_{\mathrm{A},t})}{(1-H_t)}Y$. This strategy enables the bank A to obtain the maximum degree of liquidity. Generally, the factor $\frac{1}{(1-H_t)}$ should be $\frac{1}{(1-H_t-H_{m,t})}$ for any $m \in [N] \setminus \{n\}$. We impose however this simplification since otherwise we would have to specify for each bank n the counterparty m.*

On the asset side we have to add the amount of cash obtained from the repo transactions to the liquid assets:

$$A^{\mathrm{LIQ}}_{n,t} = A^{\mathrm{LIQ}}_{n,t-1} + r \sum_{m \in D_t} L^{\mathrm{IB}}_{mn,t-1} + (1 - H_t - H_{n,t}) p^{\mathrm{C}}_t V^{\mathrm{C}}_{n,t} + \frac{(1 - H_t - H_{n,t})}{(1 - H_t)} p^{\mathrm{RR}}_t V^{\mathrm{RR}}_{n,t}, \quad t \in [T].$$

Further, we assume that initially all collateral assets are used on the repo market, i.e.

$$L^{\mathrm{R}}_{n,0} = (1 + R^{\mathrm{R}}_{n,0}) \left((1 - H_0 - H_{n,0}) p^{\mathrm{C}}_0 V^{\mathrm{C}}_{n,0} + \frac{(1 - H_0 - H_{n,0})}{(1 - H_0)} p^{\mathrm{RR}}_0 V^{\mathrm{RR}}_{n,0} \right).$$

Clearly, an increase of $R^{\mathrm{R}}_{n,t}$ increases $L^{\mathrm{R}}_{n,t}$ and an increase of H_t or $H_{n,t}$ reduces $A^{\mathrm{R}}_{n,t}$. Thus, in both cases the TTCR of bank $n \in [N]$ will decrease and forces the bank to sell even more assets.

Finally, to incorporate the interplay between the repo market and the fire sale of collateral, we assume there exists a threshold $\phi_{\mathrm{R}} \in [0,1]$ at which any bank $n \in [N]$ ceases to raise money on the repo market and instead favours a fire sale of collateral, i.e. every bank n participates in the repo market if the inequality

$$R^{\mathrm{R}}_{n,t} < \phi_{\mathrm{R}}$$

for any $t \in [T]$ is fulfilled.[9] By choosing ϕ_{R} appropriately, we can model a freeze of the repo market, since by definition no bank is willing to borrow on the repo market above this threshold.

Remark 10: Gorton and Metrick (2012) provide evidence of higher haircuts in the two weeks after Lehman Brothers' bankruptcy based on illiquidity of collateral. Haircuts for non US treasury collateral on average increased from 25% to 43% in these two weeks. As pointed out in Afonso et al. (2011), there is no evidence for hoarding behavior after Lehman Brothers' collapse and, unexpectedly, even bad performing banks did not hoard liquidity in the first days after this particular event. For that reason, we neglect this effect.

The developed setting induces the following definitions of liquidity.

Definition 13 (**Liquidity Tier I**): *We say that a bank $n \in [N]$ at time $t \in [T]$ is* liquid of tier I *if the corresponding balance sheet structure satisfies*

$$A^{\mathrm{LIQ}}_{n,t} + A^{\mathrm{IB}}_{n,t} + (1-\lambda_{n,t})\, A^{\mathrm{D}}_{n,t} + (1 - H_t - H_{n,t})\, A^{\mathrm{C}}_{n,t} - L^{\mathrm{R}}_{n,t} - L^{\mathrm{IB}}_{n,t} - \lambda_{n,t} L^{\mathrm{D}}_{n,t} > 0,$$

where $\lambda_{n,t} \in [0,1]$ is the withdrawn amount from bank n by its customers at time t.

Definition 14 (**Liquidity Tier II**): *We say that a bank $n \in [N]$ at time $t \in [T]$ is* liquid of tier II *if the corresponding balance sheet structure satisfies*

$$A^{\mathrm{LIQ}}_{n,t} + A^{\mathrm{IB}}_{n,t} + (1-\lambda_{n,t})\, A^{\mathrm{D}}_{n,t} + (1 - H_t - H_{n,t})\, A^{\mathrm{C}}_{n,t} - L^{\mathrm{R}}_{n,t} - L^{\mathrm{IB}}_{n,t} - R^{\mathrm{S}}_t L^{\mathrm{S}}_{n,t} - \lambda_{n,t} L^{\mathrm{D}}_{n,t} > 0.$$

[9] The notational convention implies that at $t = 0$ this inequality is assumed to hold.

REMARK 11: Obviously, these liquidity conditions represent a bank's ability to meet its short-term liabilities. The only difference between the *Liquidity Tier I* and the *Liquidity Tier II* condition is that the former concerns only the overnight interbank liability, whereas the latter additionally concerns the short-term (three-month) liability.

4.2.4 Asset Fire Sales

Let $\mathcal{I} = \{A^{\mathrm{C}}, A^{\mathrm{RR}}, A^{\mathrm{M}}, A^{\mathrm{CL}}, A^{\mathrm{DV}}, A^{\mathrm{F}}\}$ be the set of all available assets for each bank $n \in [N]$ which can be sold in a fire sale, and we shall use the notation $\lceil \mathcal{I} \rceil := \{\mathrm{C}, \mathrm{RR}, \mathrm{M}, \mathrm{CL}, \mathrm{DV}, \mathrm{F}\}$. In our model, the time frame for the spread of contagion is relatively short. This allows us to assume fixed interbank linkages, as banks do not have time to adapt to the threat of a financial disaster. For time frames much longer than T, the macroeconomic effects of financial contagion will extend beyond the interbank payment system, in which case our model is likely to be invalid. We adapt a discrete time framework for simplicity and assume within the short horizon under consideration, the major influential force in the financial market is the banks' behavior. Therefore, by neglecting other effects, at any time $t \in \{2, \ldots, T\}$, the market price p_t^m of any asset $m \in \lceil \mathcal{I} \rceil$ of bank $n \in [N]$ is only determined by V_t^{sold} which is the total accumulated volume of this asset sold by all banks since the beginning:

$$p_t^m = p_1^m e^{-\tau w_m \frac{V_t^{\mathrm{sold}}(m \in \mathcal{I})}{V^{\mathrm{total}}(m \in \mathcal{I})}}, \tag{4.6}$$

where p_1^m is the (randomly) drawn price at time $t = 1$ of the asset m, V^{total} is the total volume of the asset held by banks at time $t = 0$, and τ is the depreciation rate. We also assume the price impact is positively correlated with the riskiness measured by w_m in the calculation of the RWA in (4.2). In the case that fire sales occur, we take $\tau = 1.054$, since then the price drops 10% if 10% of the volume is sold, for a highly risky asset with riskiness $w_m = 1$.

Banks strive to keep their TTCR above a target threshold ϕ_{target} (cf. 4.2.1). When a negative shock occurs to its assets, a bank should sell off risky assets in exchange for the risk-free asset to improve its capital ratio.[10] More specifically, a bank immediately obtains, at time $t \in \{2, \ldots, T\}$, $\frac{p_t^m}{p_t^0} u$ units of the risk-free asset m^0 with price p_t^0 when it sells u units of a risky asset $m \in \lceil \mathcal{I} \rceil$.

[10] The risk-free asset is part of the bank $[N] \ni n$'s liquid asset $A_{n,t}^{\mathrm{LIQ}}$ at time $t \in [T]$.

ASSUMPTION 5: *If a bank $n \in [N]$ has to sell assets it always chooses to first sell those with highest risk weight, i.e. it sells an asset $m \in \mathcal{I}$ which satisfies*

$$m \in \arg\max_{l \in \mathcal{I}_n^+} w_l,$$

where $\mathcal{I}_{n,t}^+ := \{\, m \in \mathcal{I} \,|\, V_{n,t}(m) > 0\}$ and $V_{n,t}(m)$ is the volume of asset m held by bank n at time $t \in [T]$.

REMARK 12: Note that the assumption specified above does not allow a bank to hoard liquidity. This is in accordance with the findings in Afonso et al. (2011), where it is shown that there is no empirical evidence for interbank hoarding. As explained at the beginning of 4.2.2, physical interbank assets are obtained by a bank only from a defaulting counterparty, and in that case only a fraction of this interbank asset is obtained.

4.2.5 Exogenous Shocks to the Interbank System

To generate a stress scenario one has to add a shock to the system. This initial shock is a matter of choice. For example, we can choose it to be an asset price deterioration or alternatively, an increase of margins/haircuts in the repo market. In reality it is difficult the figure out which initial shocks caused a crisis, as it is also difficult to figure out which shocks occurred as a consequence of another shock. For example, a shock to the haircuts/margins on the repo market could be the consequence of an asset price shock. So the shock of haircuts/margins deserves to be termed an *indirect spill-over effect,* whereas the asset price deterioration deserves more to be called a *shock.* But the reverse scenario could also occur, i.e. an asset price shock could be the consequence of higher haircuts/margins. To overcome these difficulties, we will generally term the direct and indirect spill-over effects as shocks and let them occur all at time $t = 1$ (except fire sales, which are allowed to occur also after time $t = 1$). Otherwise we would have to define for each initial shock to the system, its impact on all other factors. This appears to be an extremely difficult task, since it is no consensus on how an initial shock impacts all other factors.

Shocks at time $t = 1$:

1. **Bank Defaults:** Defaults can occur already at time $t = 1$ and we therefore introduce the binomial random variables $D_n \sim \text{Bin}(1, p_n)$ for $n \in [N]$, taking values in $\{0, 1\}$, where $p_n = \mathbb{P}(D_n = 1)$ is the probability that a bank defaults at time $t = 1$.

2. **Asset Price Deterioration:** The return $R_t^m \in [0, 1]$ at time $t = 1$ for the asset class $m \in \lceil \mathcal{I} \rceil$ of bank $n \in [N]$ is binomial distributed, i.e. $R_t^m \sim \text{Bin}(1, q_i)$ for $i \in \{1, \ldots, 6\}$, and takes values in $\{r_1^m, r_2^m\}$ for some $r_1^m, r_2^m \in [-1, 1]$ with $r_1^m < r_2^m$, where $q_i = \mathbb{P}(R_t^m = r_2^m)$.

Liquidity shocks at time $t = 1$:

3. **Repo Rate Fluctuation Risk:** The repo rate $R_{n,t}^{\mathrm{R}}$ is randomly drawn from a binomial distribution, i.e. $R_{n,t}^{\mathrm{R}} \sim \text{Bin}(1, s_n)$, taking values in $\{r_{n,1}^{\mathrm{R}}, r_{n,2}^{\mathrm{R}}\}$ for some $r_{n,1}^{\mathrm{R}}, r_{n,2}^{\mathrm{R}} \in [0, 1]$ with $r_{n,1}^{\mathrm{R}} < r_{n,2}^{\mathrm{R}}$, where $s_n = \mathbb{P}(R_{n,t}^{\mathrm{R}} = r_{n,2}^{\mathrm{R}})$.

4. **Higher Haircuts:** Aggregate and bank specific haircuts H_t and $H_{n,t}$ for the overnight liabilities are drawn at time $t = 1$ from a binomial distribution, i.e. $H_{n,t} \sim \text{Bin}(1, v_n)$, taking values in $\{h_n, h_n'\}$ for some $h_n, h_n' \in [0, 1]$ with $h_n < h_n'$, where $v_n = \mathbb{P}(H_{n,t} = h_n')$. Similarly, $H_t \sim \text{Bin}(1, v)$ taking values in $\{h, h'\}$ for some $h, h' \in [0, 1)$ with $h < h'$, where $v = \mathbb{P}(H_t = h')$. Additionally, we require that $h_n' + h' \leq 1$ must hold.

5. **Roll Over Risk of Short-Term Debt:** The interest rate on the short-term debt R_t^{S} is randomly drawn from a binomial distribution, i.e. $R_t^{\mathrm{S}} \sim \text{Bin}(1, w)$, taking values in $\{r_1^{\mathrm{S}}, r_2^{\mathrm{S}}\}$ for some $r_1^{\mathrm{S}}, r_2^{\mathrm{S}} \in [0, 1]$ with $r_1^{\mathrm{S}} < r_2^{\mathrm{S}}$, where $w = \mathbb{P}(R_t^{\mathrm{S}} = r_2^{\mathrm{S}})$.

The interest rates on deposits R_t^{D} and on loans R_t^{A} have very little impact on a short time horizon and therefore we set them to be constant. The following table summarizes the meaning of each random variable, the possible states they can take, and their distribution:[11]

[11] An interesting extension to the current setting could be the incorporation of bank runs.

Random Variables at Time $t = 1$	**States**	**Distribution**
Default of Bank n (D_n)	$\{0, 1\}$	$\text{Bin}(1, p_n)$
Return of Collateral Assets (R_t^{C})	$\{r_1^{\text{C}}, r_2^{\text{C}}\} \in [-1, 1]$	$\text{Bin}(1, q_1)$
Return of Reverse Repo Assets (R_t^{RR})	$\{r_1^{\text{RR}}, r_2^{\text{RR}}\} \in [-1, 1]$	$\text{Bin}(1, q_2)$
Return of Residential Mortgages (R_t^{M})	$\{r_1^{\text{M}}, r_2^{\text{M}}\} \in [-1, 1]$	$\text{Bin}(1, q_3)$
Return of Corporate Loans (R_t^{CL})	$\{r_1^{\text{CL}}, r_2^{\text{CL}}\} \in [-1, 1]$	$\text{Bin}(1, q_4)$
Return of Derivatives (R_t^{DV})	$\{r_1^{\text{DV}}, r_2^{\text{DV}}\} \in [-1, 1]$	$\text{Bin}(1, q_5)$
Return of Fixed Assets (R_t^{F})	$\{r_1^{\text{F}}, r_2^{\text{F}}\} \in [-1, 1]$	$\text{Bin}(1, q_6)$
Repo Rate ($R_{n,t}^{\text{R}}$)	$\{r_{n,1}^{\text{R}}, r_{n,2}^{\text{R}}\} \in [0, 1]$	$\text{Bin}(1, s_n)$
Aggregate Haircut (H_t)	$\{h, h'\} \in [0, 1]$	$\text{Bin}(1, v)$
Individual Haircut ($H_{n,t}$)	$\{h_n, h_n'\} \in [0, 1]$	$\text{Bin}(1, v_n)$
Short-Term Interest Rate (R_t^{S})	$\{r_1^{\text{S}}, r_2^{\text{S}}\} \in [0, 1]$	$\text{Bin}(1, w)$

Tabelle 4.1: Description of the random variables in use for $n \in [N]$ and $t = 1$.

Example 5: *Taking* $\mathbb{P}(D_n = 1) = 0$, $\mathbb{P}(R_{n,t}^{\text{R}} = 1\%) = 1$, *and* $\mathbb{P}(H_{n,t} = 0\%) = 1$ *for all* $n \in [N]$, *and setting* $\mathbb{P}(R_t^{\text{C}} = 1\%) = 1$, $\mathbb{P}(R_t^{\text{RR}} = 1\%) = 1$, $\mathbb{P}(R_t^{\text{M}} = 1\%) = 1$, $\mathbb{P}(R_t^{\text{CL}} = 1\%) = 1$, $\mathbb{P}(R_t^{\text{DV}} = 1\%) = 1$, $\mathbb{P}(R_t^{\text{F}} = -20\%) = 1$, $\mathbb{P}(R_t^{\text{S}} = 1\%) = 1$, *and* $\mathbb{P}(H_t = 0\%) = 1$, *then at time* $t = 1$ *the market is shocked only by letting the price of the fixed assets drop by* -20%.

On the other hand, if we take $\mathbb{P}(D_n = 1) = 1$ *for some* $n \in [N]$, $\mathbb{P}(R_t^{\text{C}} = 1\%) = 1$, $\mathbb{P}(R_t^{\text{RR}} = 1\%) = 1$, $\mathbb{P}(R_t^{\text{M}} = -15\%) = 1$, $\mathbb{P}(R_t^{\text{CL}} = 1\%) = 1$, $\mathbb{P}(R_t^{\text{DV}} = 1\%) = 1$, $\mathbb{P}(R_t^{\text{F}} = 1\%) = 1$, $\mathbb{P}(R_t^{\text{S}} = 1\%) = 1$, $\mathbb{P}(R_{n,t}^{\text{R}} = 1\%) = 1$, $\mathbb{P}(H_t = 40\%) = 1$, *and* $\mathbb{P}(H_{n,t} = 0\%) = 1$ *for all* $n \in [N]$, *then at time* $t = 1$ *the market is shocked by bank defaults, an aggregate haircut shock, and a price deterioration of the residential mortgages.*

4.3 Simulation Results

In the following we present the simulation results for four different scenarios. The parameter values employed in the simulation study are given in 4.2, 4.3, 4.4, 4.5, 4.6, and 4.7. To make the results replicable, we use deterministic rather than probabilistic shocks. We leave the extension to a Monte Carlo simulation as an interesting avenue of future research since it is not ad hoc clear which risk measure, e.g. mean, quantile, Value-at-Risk, expected shortfall to name a few, should be applied for the determination of the TTCR.

All scenarios have in common that at time $t = 1$ an asset price shock to the mortgage loans occurs. 4.2 is drawn by only allowing direct effects to occur, meaning that a bank either directly gets insolvent due to the asset price shock or it gets insolvent as consequence of a default of another bank. 4.3 is drawn by additionally allowing for fire sales, and in 4.4 and 4.5 we further add indirect effects/shocks upon the first scenario. As 4.2 shows, when considering only direct effects the asset price shock added causes two banks to be insolvent before and at time $t = 2$, and no further insolvencies occur afterwards. By adding only one indirect effect, i.e. fire sales, we already obtain the insolvency of 7 banks in the time frame considered (see 4.3).

4.4 Conclusion

Our model provides a unified toolbox to generate stress scenarios for a system of banks which are linked through overnight interbank lending. The strength of the model lies in the numerous shocks that can be added to the system of banks, such as asset price shocks, defaults of banks, higher haircuts/margins in repo borrowing and so forth. Although our model is comprised of only 10 banks, the extension to include more banks is straightforward. Our simulation results show that indirect effects play a major role in simulating the evolvement of a shock through a system of banks and neglecting these indirect effects underestimates the extent of a crisis.

Bank-Independent Parameter Values
$R = 0.4$
$\phi_{\text{target}} = 0.1$
$\phi = 0.07$
$\phi_{\text{R}} = 0.3$
$p_0^{\text{C}} = 17$
$p_0^{\text{RR}} = 18$
$p_0^{\text{M}} = 12$
$p_0^{\text{CL}} = 19$
$p_0^{\text{DV}} = 1$
$p_0^{\text{F}} = 3$
$H_0 = 0.01$
$R_{n,0}^{\text{R}} = 0.01 \quad \forall n \in [N]$
$R^{\text{D}} = 0.01$
$R^{\text{A}} = 0.02$
$\tau = 1.054$
$\lambda_{n,t} = 0.3 \quad \forall n \in [N], \forall t \in [T]$

Tabelle 4.2: The bank-independent parameter values employed in the simulation study.

w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	w_9
0.1	0.15	0.2	0.35	0.4	0.3	0.45	0.6	0.05

Tabelle 4.3: The risk weights employed in the simulation study.

	Bank 1	**Bank 2**	**Bank 3**	**Bank 4**	**Bank 5**	**Bank 6**	**Bank 7**	**Bank 8**	**Bank 9**	**Bank 10**
Bank 1	0	960	95	664	792	430	59	562	271	640
Bank 2	599	0	184	261	315	72	513	856	398	680
Bank 3	186	362	0	899	824	138	926	757	888	832
Bank 4	720	188	295	0	287	677	821	85	246	825
Bank 5	240	34	981	191	0	895	632	252	1	381
Bank 6	595	692	33	882	960	0	335	804	655	52
Bank 7	500	375	15	748	870	572	0	901	769	626
Bank 8	52	457	907	350	512	637	691	0	268	162
Bank 9	85	305	405	507	599	534	248	990	0	424
Bank 10	609	445	764	546	366	496	601	469	244	0

Tabelle 4.4: The interbank loan matrix employed in the simulation study.

	$V^{\mathrm{C}}_{n,0}$	$V^{\mathrm{RR}}_{n,0}$	$V^{\mathrm{M}}_{n,0}$	$V^{\mathrm{CL}}_{n,0}$	$V^{\mathrm{DV}}_{n,0}$	$V^{\mathrm{F}}_{n,0}$
Bank 1	409	459	628	255	512	202
Bank 2	328	323	733	751	362	750
Bank 3	578	313	579	712	635	459
Bank 4	399	206	793	414	722	642
Bank 5	320	709	304	255	306	260
Bank 6	718	368	577	399	437	250
Bank 7	631	441	611	240	441	670
Bank 8	398	630	300	252	514	754
Bank 9	768	711	289	634	305	337
Bank 10	554	467	285	446	449	471
V^{total}	5’103	4’627	5’099	4’358	4’683	4’795

Tabelle 4.5: The initial volume of each asset for each bank employed in the simulation study.

	$A^{\mathrm{LIQ}}_{n,0}$	$A^{\mathrm{D}}_{n,0}$	$L^{\mathrm{S}}_{n,0}$	$L^{\mathrm{D}}_{n,0}$	$L^{\mathrm{L}}_{n,0}$	$H_{n,0}$
Bank 1	9479	4417	10996	6148	7550	0.01
Bank 2	2654	3231	10083	7730	8278	0.02
Bank 3	5184	3588	12666	7496	6295	0.01
Bank 4	4464	5345	11413	7470	7398	0.03
Bank 5	14115	4430	10466	7571	8229	0.001
Bank 6	15599	5356	13953	7641	8696	0.02
Bank 7	12896	4208	11053	7266	7302	0.06
Bank 8	12133	5389	10303	7495	6908	0.02
Bank 9	13951	3622	14068	9862	6376	0.02
Bank 10	10610	5932	11989	7693	7499	0.002

Tabelle 4.6: The initial liquid– and deposit assets, short-term–, deposit–, and long-term liabilities, and bank-specific haircuts employed in the simulation study.

	D	$H_{n,1}$	$R^{\mathrm{R}}_{n,1}$	R^{S}_{1}	R^{C}_{1}	R^{RR}_{1}	R^{M}_{1}	R^{CL}_{1}	R^{DV}_{1}	R^{F}_{1}	H_1
Bank 1	0	0.0166	0.02								
Bank 2	1	0.0602	0.02								
Bank 3	0	0.0263	0.02								
Bank 4	0	0.0654	0.02								
Bank 5	0	0.0689	0.08								
Bank 6	0	0.0748	0.08								
Bank 7	0	0.0451	0.02								
Bank 8	0	0.0084	0.02								
Bank 9	0	0.0229	0.02								
Bank 10	0	0.0913	0.02								
Aggregate				0.0002	-0.02	-0.04	-0.22	-0.01	-0.23	-0.015	0.01

Tabelle 4.7: The exogenous shocks to the interbank system with indirect spillover effects.

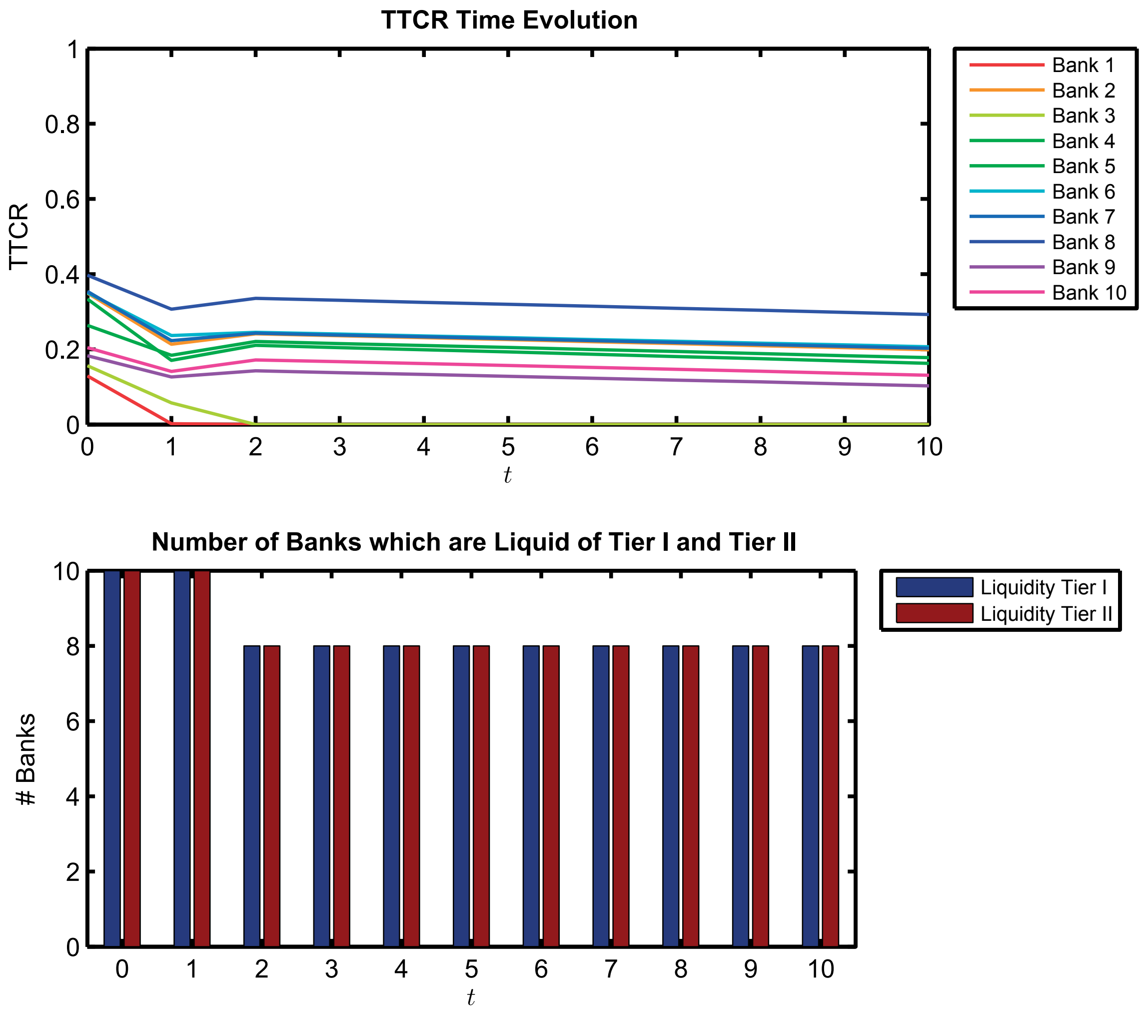

Abbildung 4.2: On the top we depict the time evolution of the TTCR for the ten different banks. Additionally, on the bottom the number of banks which satisfy the liquidity tier 1 and tier 2 measure, respectively, are presented. This illustration is produced by imposing a shock to the mortgage loans. The corresponding value can be found in 4.7. Furthermore, we do not allow for asset fire sales, i.e. $\tau = 0$.

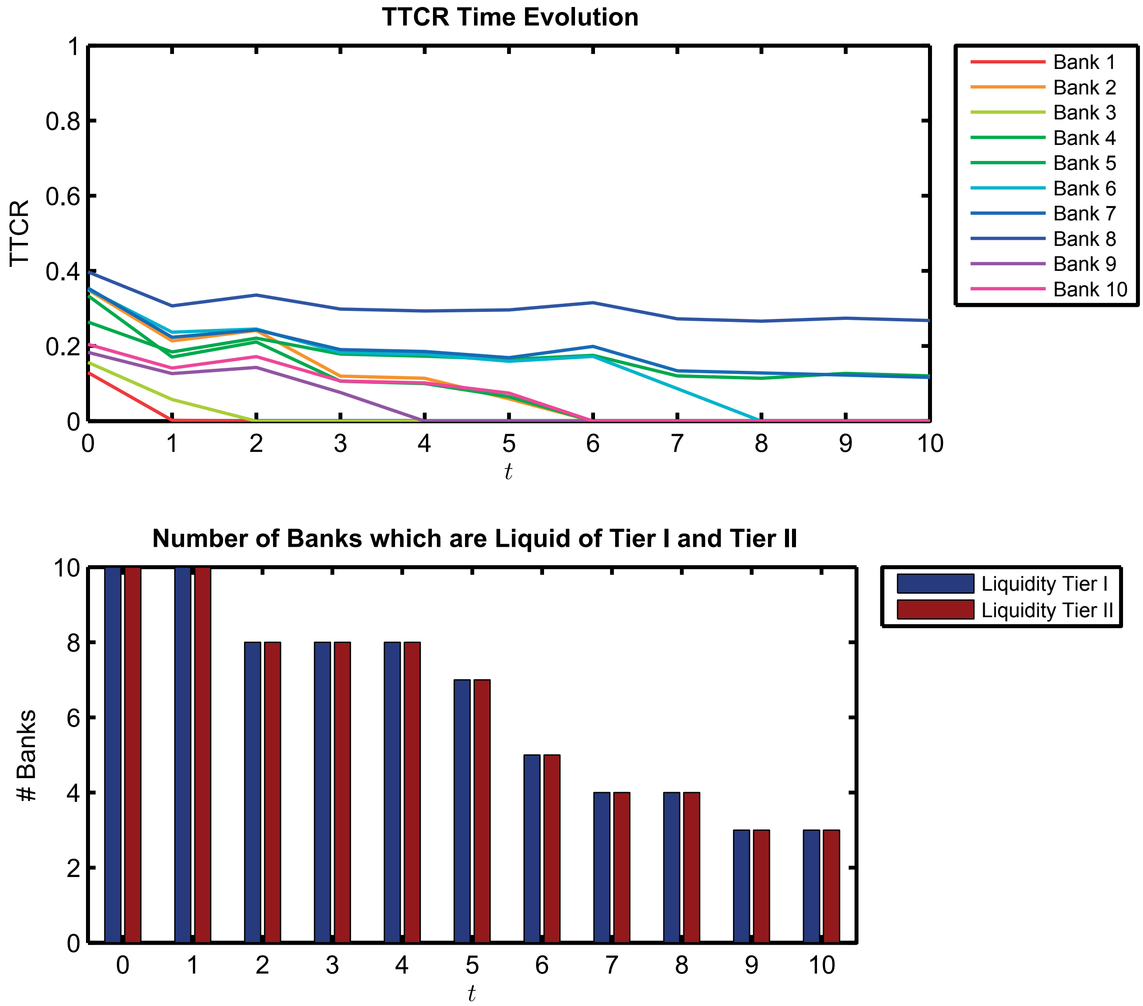

Abbildung 4.3: On the top we depict the time evolution of the TTCR for the ten different banks. Additionally, on the bottom the number of banks which satisfy the liquidity tier 1 and tier 2 measure, respectively, are presented. This illustration is produced by imposing a shock of the mortgage loans and allowing fire sales, i.e. $\tau = 1.054$. The corresponding value can be found in 4.7.

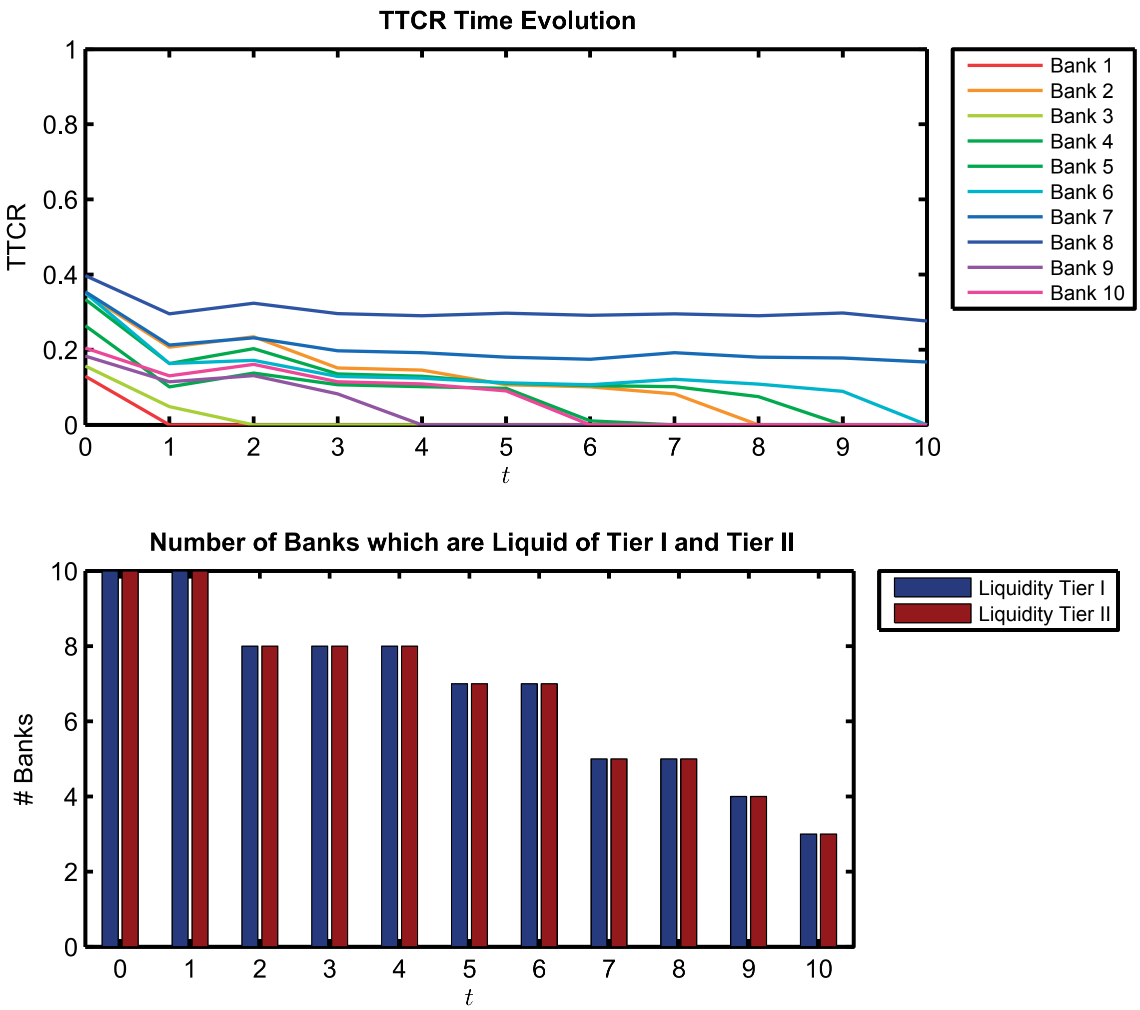

Abbildung 4.4: On the top we depict the time evolution of the TTCR for the ten different banks. Additionally, on the bottom the number of banks which satisfy the liquidity tier 1 and tier 2 measure, respectively, are presented. This illustration is produced by imposing a shock to the mortgage loans and a shock of the bank-specific repo rates and additionally allowing for fire sales, i.e. $\tau = 1.054$.. The corresponding values can be found in 4.7.

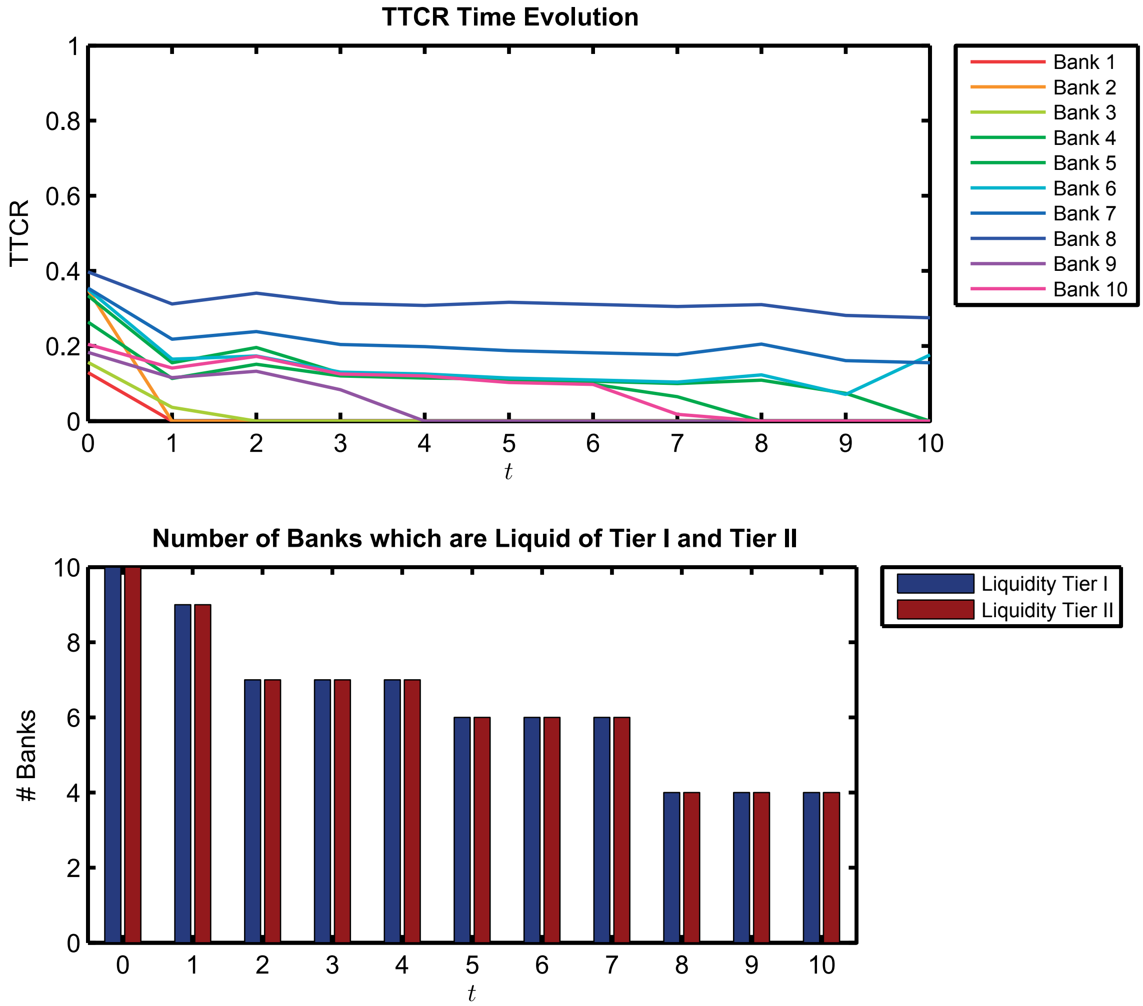

Abbildung 4.5: On the top we depict the time evolution of the TTCR for the ten different banks. Additionally, on the bottom the number of banks which satisfy the liquidity tier 1 and tier 2 measure, respectively, are presented. This illustration is produced by imposing several indirect spillover effects. The corresponding values can be found in 4.7.

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Curriculum Vitae

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